PRINCIPAL BUNDLES OVER PROJECTIVE VARIETIES

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Let X be a smooth projective variety of dimension n over $\mathbb{C}$, with a very ample line bundle $\mathcal{O}_X(1)$, and let G be a connected algebraic reductive group. A principal $\mathrm{GL}(R,\mathbb{C})$-bundle over X is equivalent to a vector bundle of rank R. If X is a curve, the moduli space was constructed by Narasimhan and Seshadri. If $\dim X > 1$, to obtain a projective moduli space we have to consider also torsion free sheaves, and this was done by Gieseker, Maruyama and Simpson. Ramanathan defined a notion of stability for principal G-bundles, and constructed the projective moduli space of semistable principal bundles on a curve.

There are two approaches to generalize Ramanathan's work to $\dim X > 1$, the work of Schmitt [Sch1, Sch2] and the work of Gómez and Sols [G-S].

In the work of Gómez and Sols, to obtain a projective moduli space we introduce *semistable principal G-sheaves*. These are triples (P, E, ψ), where E is a torsion free sheaf on X, P is a principal G-bundle on the open set U where E is locally free, and ψ is an isomorphism between $E|_U$ and the vector bundle associated to P by the adjoint representation.

We say it is (semi)stable if all filtrations $E_\bullet$ of E as sheaf of (Killing) orthogonal algebras, i.e. filtrations with $E_i^\perp = E_{-i-1}$ and $[E_i, E_j] \subset {E_{i+j}}^{\vee\vee}$, have

$$\sum (P_{E_i} \operatorname{rk} E - P_E \operatorname{rk} E_i)\,(\preceq)\,0,$$

where P_{E_i} is the Hilbert polynomial of E_i. After fixing the Chern classes of E and of the line bundles associated to the principal bundle P and characters of G, we obtain a projective moduli space of semistable principal G-sheaves. In case $\dim X = 1$, this notion of (semi)stability is equivalent to Ramanathan's notion.

In the approach of Schmitt, which works for when G is a semisimple group, we fix a faithful representation $\rho : G \to \mathrm{SL}(V)$, and define a *honest singular principal bundle* as a triple (P, E, ψ), where E and P are as before and ψ is an isomorphism between $E|_U$ and the vector bundle associated to P by the representation ρ. Note that the only difference is that, in the previous approach, the representation was the adjoint representation, but now ρ a faithful representation and G is semisimple. Schmitt gives a notion of (semi)stability, which coincides with Ramanathan's for $\dim X = 1$ and with Gomez-Sols's if ρ is the adjoint representation and the center of G is trivial (so that ρ is faithful).

In both cases, after fixing the Hilbert polynomials of the torsion free sheaves, one constructs a projective moduli space of semistable objects. In these lectures we will discuss both constructions. If time permits, I will also discuss some generalizations to principal G-bundles of results known for vector bundles (Grauert-Mülich restriction theorems [B-G],...).

References

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[Sch1] A. Schmitt, *Singular principal bundles over higher-dimensional manifolds and their moduli spaces,* Internat. Math. Res. Notices **23** (2002), 1183–1210. `math.AG/0201085`

[Sch2] A. Schmitt, *A closer look at semistability for singular principal bundles* to appear in Internat. Math. Res. Notices `math.AG/0405417`