ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO

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Andrzej Zajtz

Primitive geometric objects

Introduction. Let $\Omega = (X, \Phi)$ be a geometric object with the fibre X (the set of values) and the transformation formula Φ[1], i.e. the corresponding transformation group G of local coordinate system acts on X as follows

(1) $$X \times G \ni (x, g) \to x' = \Phi(x, g)\,.$$

An object $\theta = (Y, \Psi)$ is called a concomitant of Ω if there exists a mapping

(2) $$h\colon\ X \to Y$$

of X onto Y which is invariant under transformations of fibres X, Y, i.e. it satisfies the condition

(3) $$h\big(\Phi(x, g)\big) = \Psi\big(h(x), g\big)$$

for arbitrary x, g.

A concomitant θ is called *trivial* if it is a constant or θ is equivalent to Ω, a *proper concomitant* in the opposite case and *scalar non-trivial* if θ is a scalar with at least two different values.

A geometric object will be called *primitive* if it does not exist any its proper concomitant. A primitive geometric object will be called a *strictly primitive* object if it has not any scalar non-trivial concomitant.

In the present paper we deal with a problem of primitivity of geometric objects in general. Then we prove this property for some objects of the first class and we give the complete classification of the primitive objects of the second class.

1. The strictly primitivity of a primitive geometric object is closely bounded with the transitivity of this object. This follows from the following proposition.

[1] This representation of a geometric object is closely connected with the notion of an abstract geometric object in sense of Kucharzewski-Kuczma [4] or with the definition of Haantjes-Laman (cf. also [4]).

Proposition 1. *A geometric object has a non-trivial scalar concomitant if and only if it is intransitive.*

Proof. The condition (3) for a scalar concomitant θ of a given object Ω takes the following form

$$h\big(\Phi(x, g)\big) = h(x) .$$

If we fix x then this equality holds for any $g \in G$, i.e. the domain of $h(x)$ is the whole set $\Phi(x, G)$ that is a transitive fibre of object Ω which passes trough x. Thus there exist two different values $h(x), h(x')$ if and only if Ω has at least two different transitive fibres, i.e. if it is an intransitive object. On the other hand, if Ω is intransitive, then any mapping (2) which is constant on every transitive fibre and takes at least two different values, represents a non-trivial scalar concomitant of Ω.

From the above proposition we get immediately the following

Proposition 2. *A primitive geometric object is strictly primitive if and only if it is transitive.*

We omit the simple proofs of the following propositions 3 and 4.

Proposition 3. *If we have the two equivalent objects, then they both are primitive, or both are imprimitive.*

Proposition 4. *An object is primitive if and only if each its transitive fibre (that means an independent geometric object with the same transformation formula) is primitive.*

In virtue of the above property we can reduce the classification of primitive objects to the determination of primitive transitive objects.

Another form of primitivity represents so called simple geometric objects defined by V. V. Vagner [6]. Vagner calls a differential geometric object of the class r a simple geometric object, if there do not exist concomitants that are geometric differential objects of the class $<r$ and >0.

He gives too the full classification of simple objects of the class $r > 1$ (cf. [6]). Vagner proves for instance that there do not exist simple differential geometric objects of the class $r > 2$ in n-dimensional space if $n > 1$. For $n = 1$ the same proposition holds for objects of the class $r > 3$. Since every primitive geometric object is also simple, then we get immediately from Vagner's results the following proposition.

Proposition 5. *There are no primitive differential geometric objects of the class $r > 2$ if $n > 1$ and of the class $r > 3$ if $n = 1$*[1].

2. Primitive objects and maximal subgroups

Now, we shall give a group-theoretical condition for the primitivity of transitive geometric objects. As a matter of fact it will be the same condition as for the primitivity of a representation of a group.

[1] In fact, Vagner has classified only the regular geometric objects (the transformation formula Φ is analytic.)

Let be given an object (1). We denote by H the stationary subgroup of a fixed point x of X, i.e. H is the set of all the elements $g \in G$ which leave the point x invariant. The object (X, Φ) as a structure is isomorph to the homogeneous space G/H in which G acts as the left translations group.

Let (Y, Ψ) be a concomitant of (X, Φ) and h a mapping (2). If K denote the stationary subgroup of image $h(x)$ then evidently the following relation holds

$$H \subset K \subset G. \tag{4}$$

A concomitant (Y, Ψ) determines in the above manner a subgroup K satisfying condition (4) for any fixed subgroup H.

Conversely, let K be a subgroup satisfying (4) for a fixed H. It is well-known (cf. [2], p. 109) that the structure of homogeneous space G/K is isomorph to the factor-structure $(G/H)/(K/H)$. Let (Y, Ψ) be an object which is isomorph to G/K, thus also to the above mentioned structure. It follows from the following diagram, that (Y, Ψ) is a homomorph image of the structure (X, Φ), i.e. (Y, Ψ) is a concomitant of the object $\Omega = (X, \Phi)$.

$$\begin{array}{ccccc} & & G/H & \leftrightarrow & (X, \Phi) \\ & & \downarrow & & \\ (G/H)/(K/H) & \leftrightarrow & G/K & \leftrightarrow & (Y, \Psi) \end{array}$$

(„$\rightarrow$"—homomorphism).

Thus we have proved that any subgroup K satisfying the relation (4) determines (with accuracy to the equivalence) a concomitant of given object (1).

If $K = H$ then the corresponding concomitant is equivalent with the object Ω; if $K = G$ then it is a trivial scalar, while the space G/G consists of one point only. Thus (Y, Ψ) is a proper concomitant of Ω if and only if the subgroup K satisfies except the relation (4) also the condition

$$K \neq H \quad \text{and} \quad K \neq G. \tag{5}$$

It follows from the above results that an object (1) has a proper concomitant if and only if its stationary subgroup H [1] is maximal, i.e. if there do not exist subgroups K of G satisfying the relations (4) and (5).

As follows by using the word "primitive" we can that formulate:

Theorem 1. *A transitive geometric object is primitive if and only if its stationary subgroup is maximal in the transformation group G.*

The analogous condition for intransitive objects from that and from the proposition 4:

Theorem 2. *An intransitive geometric object is primitive if and only if a stationary subgroup of any of its points is maximal.*

We shall use these conditions practically in the next sections.

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[1] Defined with occuracy to the interior authomorphism of G.

3. The primitivity of some linear objects of the first class

A differential geometric object of the type (m, n, r) is mentioned (classically) as an object (X, Φ) such that: (a) X is a subset of R^m (m - number of components) and (b) $G = L_n^r$ — the group of all jets of order r of differomorphisms in R^n at the origin.

The condition (a) leads sometimes to a non-consequence, if we consider for instance a so-called Pensov object ω with the transformation formula

$$\omega' = \frac{A_1^{1'} + A_2^{1'}\omega}{A_1^{2'} + A_2^{2'}\omega}, \qquad [A_j^i] \in L_n^1 .$$

It is treated as a differential geometric object of the type (1, 2, 1), although its set of values X is a one-dimensional projective space P.

Some extension of condition (a) was given by E. Siwek [5]. Instead of (a) he assumes that X is a subset of some factor-space R^m/S of space R^m by an equivalence relation S defined on R^m. In such a manner generalized objects Siwek calls the pseudoobjects. (The notion which has been used earlier by some mathematitians for some objects, without any strict definition.) These objects are in fact the factor-objects in sense of factor-structure of structure (X, Φ) by the relation S.

The fact that in every case of an pseudoobject we ought to mention the corresponding equivalence relation is not convenient. Therefore we deal with the general situation and we shall understand an object (X, L_n^r) by a differential geometric object, where X is a subset of R^m or P^m and in the last case we identify the objects and pseudoobjects.

Let $\Omega = (\omega_1, \ldots, \omega_m)$ (ω_i—real components) be a linear homogeneous geometric object. We put

$$\vartheta_j = \frac{\omega_i}{\omega_1}, \qquad (j = 2, \ldots, m). \tag{6}$$

The numbers ϑ_j represent a (non-linear) geometric object; it will be called a *projective object* associated with the object Ω and denoted by Π. Its fibre is a subset of $m-1$ dimensional projective space P^{m-1}. The components ω_i $(i = 1, \ldots, m)$ can be regarded as homogeneous components of Π.

For example, the above mentioned Pensov object ω can be obtained as a projective object, associated with the contravariant vector $v = (v^1, v^2)$, such that $\omega = v_2/v_1$. The numbers v^1, v^2 represent here the homogeneous components of the object ω.

Let

$$v^{i_1,\ldots,i_p} = \underset{1}{v}^{[i_1}\underset{2}{v}^{i_2}\ldots\underset{p}{v}^{i_p]} \qquad (i = 1, \ldots, n) \tag{7}$$

be a simple p-vector ($p \geqslant 1$). We shall study now the corresponding projective object (6) which will be called a *projective p-vector*. Not writing exactly its components we shall regard the number (7) as the homogeneous components of this object (according to the above remark).

To every homogeneous system of components (7) we have a one-to-one correspondence between a linear subspace $V^p \subset V^n$ spanned on the vectors $\underset{1}{v}, \ldots, \underset{p}{v}$ from (7). We thus can consider the components (7) as the homogeneous Plücker coordinates of p-dimensional subspaces of V^n. Consequently there exists a one-to-one correspondence between the set of values X of the projective p-vector and the Grassmanns manifold $M(p, n)$ of all the p-subspaces of V^n and moreover the structure of this object is isomorph with the structure of $M(n, p)$ under the transformations of the linear group $GL(n)$. In particular the transitivity of the considered object follows from the above.

Let us choose the point π_0 of X such that the corresponding subspace V^p is spanned on the first p unit vectors $e_1, \ldots, e_p$. Then the stationary subgroup of π_0 (nb. leaving this V^p invariant) consists of all the matrices of the form

$$\begin{bmatrix} a_{11} & \ldots & & & a_{1n} \\ \cdot & & & & \\ \cdot & & & & \\ a_{p1} & \ldots & a_{pp} & \ldots & \\ 0 & & 0 & a_{p+1,p+1} & \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ 0 & & 0 & a_{n,p+1} & \ldots \end{bmatrix}$$

Lemma 1. *The subgroup H of all the matrices* (8) *is a maximal subgroup of $GL(n)$.*

Proof. The subgroup $H' \subset H$ of all unimodular matrices of the form (8) is maximal in $SL(n)$, while it consists of all unimodular matrices leaving invariant a linear subspace (cf. Dynkin [3]). Each subgroup K containing H and different from it must have at least one non-zero element on a zero place in (8); its unimodular subgroup K' contains H' and $K' \neq H'$: From the maximality of H' follows $K' = SL(n)$. On the other hand, the determinants of matrices of K can take arbitrary values; consequently it must hold $K = GL(n)$. This completes the proof of the lemma.

In virtue of the above lemma and of the theorem 1 we get immediately the following

Theorem 3. *Every projective simple p-vector* $(1 \leqslant p \leqslant n)$ *is strictly primitive.*

In particular the projective 1-vector (i.e. a projective vector) is a Pensov object and thus we get

Corollary 1. *The Pensov object is strictly primitive.*

4. Primitive objects of the second class

The group $G = L_n^2$ consists of all the pairs (a, x) where a is arbitrary non-singular matrix of order n, i.e. $a \in GL(n)$ and $x = (x_{jk}^i)$ arbitrary system of $n\binom{n+1}{2}$ numbers with symmetry condition $x_{jk}^i = x_{kj}^i$ $(i, i, k = 1, \ldots, n)$.

The group product $(c, z) = (a, x)(b, y)$ is defined as follows

$$c = ab\,, \qquad z^i_{jk} = x^i_{pq} b^p_j b^q_k + a^i_p y^p_{jk}\,. \tag{9}$$

Let us denote the two following characteristic subgroups of G and namely

$$G^0 = \{(a, 0),\ a \in GL(n)\}$$

$$G' = \{(e, x),\ e\text{—unit matrix},\ x \text{ arbitrary}\}$$

G' being an abelian group $\big((e, x)(e, y) = (e, x+y)\big)$.

Lemma 2. *Let V be a vector space of all the elements $x^i_{jk} (= x^i_{kj})$ on which the linear group $GL(n)$ acts as follows*

$$\bar{x}^i_{jk} = a^{-1i}_p x^p_{qr} a^q_j a^r_k\,. \tag{10}$$

The unique non-trivial subspace of V which are invariant under transformation (10) *are the following*

$$W\colon\ x^p_{pk} = 0\,, \qquad Z\colon\ x^i_{jk} - \frac{1}{n+1}\delta^i_{(j} x^p_{k)p} = 0\,, \tag{11}$$

$$\textit{for } i, j, k = 1, \ldots, n\,.$$

The above lemma follows immediately from the corresponding Vagner's results in (6).

Lemma 3. *The unique subgroups of G satisfying the condition*

$$H^0 \overset{\mathrm{df}}{=} H \cap G^0 = G^0 \tag{12}$$

i.e. the intersection of a given subgroup H with G^0 is G^0, are the subgroups A and B with the elements (a, x) satisfying the relations

$$A\colon\ a^{-1p}_q x^q_{pk} = 0\,, \qquad (k = 1, \ldots, n)\,, \tag{13}$$

$$B\colon\ a^{-1j}_p x^p_{jk} - \frac{1}{n+1} a^{-1q}_p \delta^i_{(j} x^p_{k)q} = 0\,, \qquad (i, j, k = 1, \ldots, n)\,. \tag{14}$$

Proof. Let H be a subgroup satisfying condition (12). Let us denote by H' the intersection H with G' (i.e. an invariant subgroup of G). H' is an invariant subgroup in H, namely if $(a, x) \in H$ then we have $(a, x)^{-1}(e, u)(a, x) = (e, w)$ where

$$w^i_{jk} = a^{-1i}_p u^p_{qr} a^q_j a^r_k \tag{15}$$

and $(e, w) \in H \cap G' = H'$. On the other hand H' is a subspace of vector space G'. In virtue of (15) H' is a subspace invariant under transformation (15) (cf. (10)). It follows from the assumption (12) that the matrices a^i_j in (15) are arbitrary matrices of $GL(n)$. According to lemma 2 H' is either the subspace A' consisting of pairs (e, x) where $x \in W$, or the subspace B' of pairs (e, y) where $y \in Z$ (W, Z as in (11)).

Let A be a subgroup containing A' and satisfying (12). The intersections $A^0 = A \cap G^0 = G^0$ and $A' = A \cap G'$ determine the subgroup A completely (analogously for the corresponding subgroup B): In fact, if $(a, x) \in A$ then also $(a^{-1}, 0)(a, x) = (e, u)$ where

(16) $$u^i_{jk} = a_p^{-1i} x^p_{jk}$$

belongs to A. Moreover (e, u) belongs to A', thus $u \in W$ and from (11) and (16) we get the relations (13). Conversely, it is easy to see that all the elements (a, x) which satisfy (13) form a subgroup of G such that the intersection with G gives G and this with G' gives A'. By a similar way we determine the subgroup B.

Lemma 4. *The subgroups A, B from lemma 3 are the unique maximal subgroups of G satisfying the condition*

(14) $$H \cap G' \neq G'\,^1.$$

Proof. If a subgroup H is maximal in G then it must satisfy condition (12), otherwise in view of (17) H would be contained in the subgroup K for which

$$K \cap G^0 = H^0 \quad \text{and} \quad K \cap G' = G'.$$

Since $H^0 \neq G^0$ so $K \neq G$ and we get a contradiction to maximality of H. Thus it must be $H^0 = G^0$ and according to lemma 3 H is either the subgroup A or B, q.e.d.

Theorem 4. *Every strictly primitive differential geometric object of the second class is equivalent either to the contracted object of the affine connection $\Gamma_k = \Gamma^p_{pk}$ or to the object of projective connection $\Pi^i_{jk} = \Gamma^i_{jk} - \frac{1}{n+1}\delta^i_{(j}\Gamma_{k)}$.*

Proof. The stationary subgroups of the zero-elements of the above mentioned objects Γ_k and Π^i_{jk} are exactly the subgroups A and B from lemma 3, respectively. In view of lemma 4 these are the unique maximal subgroups of G satisfying condition (17). Thus in virtue of theorem 1 there do not exist other (with occuracy to the equivalence) strictly primitive objects of the second class which are different from the above.

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[1] The stationary subgroups of the geometric objects, which are of the second class essentially, must satisfy this condition, otherwise these objects are of the first class.