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On the Dirichlet's problem for a class of the elliptic systems of differential equations of the second order

1. Let D be a bounded domain in the space of m variables $X = (x_1, \ldots, x_m)$. Consider the system of equations

$$\sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{m} b_k(X)\frac{\partial u}{\partial x_k} + C(X)u = f(X)\,, \tag{1}$$

where $a_{ij}(X) = a_{ji}(X)$, $b_k(X)$ $(i, j, k = 1, \ldots, m)$ are defined and bounded in the closure $\bar{D}$ of D. Let the characteristic form $\sum\limits_{i,j=1}^{m} a_{ij}(X)\lambda_i\lambda_j$ be uniformly positive definite in $\bar{D}$ and let the elements $c_{ij}(X)$ $(i, j = 1, \ldots, n)$ of the matrix $C(X) = \{c_{ij}(X)\}$ be defined and bounded in $\bar{D}$. The vectorial function $f(X) = \{f_i(X)\}$ $(i = 1, \ldots, n)$ is defined and bounded in $\bar{D}$.

If the vectorial function $u(X) \neq 0$ in D, then putting

$$R(X) = [u_1^2(X) + \ldots + u_n^2(X)]^{1/2}$$

we have $u(X) = R(X)e(X)$, where $e(X)$ is the unit vector parallel to $u(X)$.

It is easy to verify that the function $R(X)$ fulfils in D the equation

$$\sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial^2 R}{\partial x_i \partial x_j} + \sum_{k=1}^{m} b_k(X)\frac{\partial R}{\partial x_k} + \left(e^* C(X)e - \sum_{i,j=1}^{m} a_{ij}\frac{\partial e^* \partial e}{\partial x_i \partial x_j}\right)R = e^* f(X)\,. \tag{2}$$

By the assumptions on the coefficients of (1), one may apply to equation (2) the extremum property [2] in the following form:

If the assumptions on the coefficients of system (1) are satisfied, the quadratic form

$$\sum_{i,j=1}^{n} c_{ij}(X)\lambda_i\lambda_j \tag{3}$$

is negative definite in $\bar{D}$, and if $e^*f(X) \geqslant 0$ in $\bar{D}$, then the function $R(X)$ does not attain its supremum in D, provided $R(X) \neq \text{const}$.

To equation (2) may be applied also the Olejnik's lemma [3]; according to this lemma we have the following

Theorem 1. *If the boundary $F(D)$ of D or at least a closed part Σ of $F(D)$ is of class C^2 (i.e. $F(D)$ or Σ is given by the equation $G(X) = 0$, where the function G is of class C^2 in an m-dimensional domain containing $F(D)$ or Σ and, moreover, $\operatorname{grad}^2 G > 0$), $u^*(X)f(X) \leqslant 0$ (or $u^*(X)f(X) \geqslant 0$), a part of $F(D)$ contained in a neighbourhood of a point X_0 belong to Σ, the quadratic form (3) is negative definite in $\bar{D}$, $u(X)$ is a regular solution of system (1) (i.e. $u(X)$ is of class C^2 in D and continuous in $\bar{D}$), $R(X_0) > 0$ (or $R(X_0) = 0$) and $R(X) < R(X_0)$ (or $R(X) > R(X_0)$) for $X \in D$, then*

$$\varlimsup_{X\to X_0} \frac{R(X)-R(X_0)}{\overline{XX_0}} < 0 \qquad \left(or \varliminf_{X\to X_0} \frac{R(X)-R(X_0)}{\overline{XX_0}} > 0\right),$$

as $X \in D$ tends to X_0 along a half-straightline which goes from X_0, and is not tangent to $F(D)$ at X_0 and is contained in neighbourhood of X_0.

Proof. The assertion of theorem 1 in the case $R(X_0) > 0$ follows from the Olejnik's lemma applied to equation (2). In the case $R(X_0) = 0$ the coefficient

$$\bar{c}(X) = e^*C(X)e - \sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial e^*}{\partial x_i}\frac{\partial e}{\partial x_j}$$

and the function $\bar{f}(X) = e^*f(X)$ are not defined at the point X_0. Since $\bar{c}(X) \leqslant 0$ for $X \in \bar{D}$ and $X \neq X_0$, then putting $\bar{c}(X_0) = 0$ we have $\bar{c}(X) \leqslant 0$ for $X \in \bar{D}$. Alike, we define $\bar{f}(X_0) = 0$, then $\bar{f}(X) \leqslant 0$ for $X \in \bar{D}$. Then the functions $\bar{c}(X)$ and $\bar{f}(X)$ satisfy the assumptions of the Olejnik's lemma, from which follows the thesis of theorem 1.

Remark 1. The second part of theorem 1 (i.e. in the case $R(X_0) = 0$) is also true without the assumption relative to the definiteness of the form (3). Namely we have the following

Theorem 2. *Under the assumptions of theorem 1, if $R(X_0) = 0$, then the second part of the assertion of theorem 1 is satisfied even if the form (3) is not definite.*

Proof. By the transformation

$$u(X) = w(X)v(X)\,, \quad w(X) = \prod_{i=1}^{m} \cos\mu(x_i - \overset{0}{x}_i) \qquad X_0 = (\overset{0}{x}_1, \ldots, \overset{0}{x}_m) \tag{4}$$

(1) takes the form (for $w(X) \neq 0$)

$$\sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{k=1}^{m} \tilde{b}_k(X)\frac{\partial v}{\partial x_k} + \tilde{C}(X)v = \tilde{f}(X) \tag{5}$$

where

$$\tilde{f}(X) = \frac{1}{w(X)} f(X), \quad \tilde{C}(X) = \frac{1}{w} L(w) E + C(X),$$

$$L(w) = \sum_{i,j=1}^{m} a_{ij}(X) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{k=1}^{m} b_k(X) \frac{\partial w}{\partial x_k}$$

and E denotes the unit matrix of the rank n.

We shall show that there exists a number μ and a neighbourhood $0(X_0)$ of X_0 such that the quadratic form

$$\sum_{i,j=1}^{n} \tilde{c}_{ij}(X) \lambda_i \lambda_j$$

is uniformly negative definite and $w(X) > 0$ in $0(X_0) \cap \bar{D}$. Indeed, put

(6) $$M(X) = \sup_{|\Lambda|=1} \sum_{i,j=1}^{n} c_{ij}(X) \lambda_i \lambda_j \qquad \Lambda = (\lambda_1, \ldots, \lambda_n).$$

It is sufficient to choose a number μ and $0(X_0)$ such that for $X \in 0(X_0) \cap \bar{D}$

(7) $$L(w) + M(X) w < 0 \quad \text{and} \quad w(X) > 0.$$

The proof of inequality (7) is done in [1]. Then the system of equations (5) satisfies the assumptions of theorem 1, from which we have

(8) $$\lim_{X \to X_0} \frac{\bar{R}(X) - \bar{R}(X_0)}{\overline{XX_0}} > 0,$$

where $X \to X_0$ along a half-straightline going from X_0, not tangent to $F(D)$ at X_0 and contained in $O(X_0) \cap D$,

$$\bar{R}(X) = [v_1^2(X) + \ldots + v_n^2(X)]^{1/2}.$$

On the other hand, since $\bar{R}(X_0) = R(X_0) = 0$ and $R(X) = w(X) \bar{R}(X)$,

(9) $$\frac{R(X) - R(X_0)}{\overline{XX_0}} = \frac{\bar{R}(X) - \bar{R}(X_0)}{\overline{XX_0}} w(X).$$

Observe that

(10) $$\lim_{X \to X_0} w(X) = 1.$$

The assertion of theorem 2 now follows from (8), (9) and (10).

The following corollaries are simple consequences of theorem 2

Corollary 1. *If $u(X)$ is the solution of the system* (1), *such that* $u(X_0) = 0$, $u(X) \neq 0$ *for* $X \in 0(X_0)$ *and* $X \neq X_0$ $(X_0 \in D)$, *then* $\sum_{i=1}^{n} \operatorname{grad}^2 u_i > 0$ *at* X_0.

Proof. Suppose $\sum_{i=1}^{n} \operatorname{grad}^2 u_i = 0$ at X_0. Since the functions $\frac{\partial u}{\partial x_i}$ are continuous at X_0 (for $X_0 \in D$), then $\frac{du}{dl} = 0$ for each direction l, which goes from X_0, i.e.

$$\lim_{X \to X_0} \frac{u(X) - u(X_0)}{\overline{XX_0}} = 0 . \tag{11}$$

On the other hand, since $R(X_0) = u(X_0) = 0$, so

$$\frac{R(X) - R(X_0)}{\overline{XX_0}} = \frac{u(X) - u(X_0)}{\overline{XX_0}} e^*(X);$$

from this and (11) we have

$$\lim_{X \to X_0} \frac{R(X) - R(X_0)}{\overline{XX_0}} = 0 ,$$

in contradiction with the assertion of theorem 2.

Corollary 2. *If $u(X)$ is a biregular solution of the system* (1) (*i.e.* $u(X)$ *is of class C^2 in D and of class C^1 in $\bar{D}$*), $u(X_0) = 0$, $\sum_{i=1}^{n} \operatorname{grad}^2 u_i = 0$ at X_0, *there exists a ball $K(X_0)$ whose boundary passes through X_0 and whose other points all belong to D, then the function $u(X)$ cannot be different from zero in none neighbourhood of X_0.*

2. If the quadratic form (3) is negative definite, then to the system (1) one may apply the extremum property. From this property, as in the case of one equation, follows the uniqueness of the solution of Dirichlet's problem for the system (1) in the domain D. Sometimes, as in the case of one equation, it is possible to assure the uniqueness of the solution of Dirichlet's problem also in the case when the quadratic form (3) is not negatively definite in the domain $\bar{D}$. We shall prove the following

Theorem 3. *If there exists a function $w(X)$ regular in $\bar{D}$ and if $w(X)$ satisfies the conditions*

$$w(X) > 0 \quad \textit{for} \quad X \in \bar{D}, \tag{12}$$

$$L(w) + M(X) w < 0 \quad \textit{for} \quad X \in \bar{D}, \tag{13}$$

where $M(X)$ is defined by the formula (6), *then the only solution of the homogeneous Dirichlet's problem for the system* (1) *is the function $u(X) \equiv 0$.*

Proof. The transformation

$$u(X) = w(X) v(X) .$$

gives for the function $v(X)$ the following system

$$\sum_{i,j=1}^{m} a_{ij}(X) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{k=1}^{m} \tilde{b}_k(X) \frac{\partial v}{\partial x_k} + \tilde{C}(X) v = 0 , \tag{14}$$

where $\tilde{C}(X) = \frac{1}{w} L(w)E + C(X)$. Due to (12) and (13) the quadratic form

$$\sum_{i,j=1}^{n} c_{ij}(X)\lambda_i\lambda_j$$

is negative definite in $\bar{D}$. Moreover, the function $v(X)$ is regular in $\bar{D}$ and vanishes on the boundary $F(D)$ of D, so that from the extremum property $v(X) \equiv 0$, and therefore $u(X) \equiv 0$ in $\bar{D}$.

3. Evidently, the theorems on the uniqueness of the solution of the Dirichlet's problem for the system (1), as in the case of one equation, in general are not true in the case of an unbounded domain D. Therefore, we shall now prove the following

Theorem 4. *If the coefficients $a_{ij}(X), b_k(X)$ $(i, j, k = 1, \ldots, m)$, and the elements of the matrix $C(X)$ are defined and bounded in an unbounded domain $\bar{D}$, there exists a regular function $w(X)$ in $\bar{D}$ such that $w(X) > 0$ and $L(w) + M(X)w < 0$ for $X \in \bar{D}$, then in the class of all regular functions in $\bar{D}$ satisfying the condition*

$$\lim_{\overline{0X}\to\infty} \frac{R(X)}{w(X)} = 0, \qquad X \in D \tag{15}$$

$\left(\text{where } \overline{0X} = \sqrt{\sum_{i=1}^{m} x_i^2}\right)$ the function $u(X) \equiv 0$ is the only solution of the homogeneous Dirichlet's problem for the system (1).

Proof. Let $u(X) = w(X)v(X)$. The function $v(X)$ satisfies the system (14) in D and $v(X) = 0$ on $F(D)$. Denote $\Delta_\varrho = D \cap K_\varrho$, where K_ϱ is a ball of centre at the origin and of the radius ϱ. Due to (15) let us choose for a number $\varepsilon > 0$ a number ϱ such that $\bar{R}(X) < \varepsilon$ for $X \in F(K_\varrho) \cap \bar{D}$. Since $\bar{R}(X) = 0$ on $F(D)$, then $\bar{R}(X) < \varepsilon$ for $X \in F(\Delta_\varrho)$. But from the extremum property applied to the system (14) we have $\bar{R}(X) < \varepsilon$ for $X \in \Delta_\varrho$. If X_0 is an arbitrary point of D, we choose a number ϱ such that $X_0 \in \Delta_\varrho$. Then $\bar{R}(X_0) < \varepsilon$. Since ε is arbitrary, so $\bar{R}(X_0) = 0$, then $v(X_0) = 0$, hence $u(X_0) = 0$, then $u(X) \equiv 0$ in the domain $\bar{D}$. Using theorem 4 we shall prove the following

Theorem 5. *If the coefficients of the system* (1) *satisfy the assumptions of theorem* 4 *and if*

$$\sum_{i,j=1}^{n} c_{ij}(X)\lambda_i\lambda_j \leqslant -c_0 \sum_{i=1}^{n} \lambda_i^2$$

for $X \in \bar{D}$, where $c_0 > 0$, then in the class of the functions regular in $\bar{D}$ and satisfying the condition $R(X) \leqslant M\varrho^p$ $\left(\varrho = \sqrt{\sum_{i=1}^{m} x_i^2}\right.$, p is an arbitrary positive integer), there is the uniqueness of solutions of the Dirichlet's problem for the system (1) *in the domain D.*

Proof. We shall prove that the only solution of the homogeneous Dirichlet's problem for the system (1) is the function $u(X) \equiv 0$. For this purpose