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A sharp estimation of a_5 in a subclass of the class S

1. **Jenkins** proved [4] the following theorem:

If $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ *is analytic and univalent in* $|z| < 1$, *i.e.* $f(z) \in S$, *and if* $a_j = 0$ *for* $j < n$, *n—arbitrary fixed natural number, then for* $n \geqslant 2$

$$(1)\qquad |a_{2n-1} - \beta a_n^2| \leqslant \begin{cases} \dfrac{4\beta - n - 1}{(n-1)^2} & \text{for} \quad \beta \geqslant \dfrac{n}{2}, \\ (n-1)^{-1}\left[1 + 2\exp\left(-2\dfrac{n+2\beta-2}{n-2\beta}\right)\right], & \dfrac{n}{2} > \beta > 1 - \dfrac{n}{2}, \\ \dfrac{1+n-4\beta}{(n-1)^2} & \text{for} \quad \beta \leqslant 1 - \dfrac{n}{2}. \end{cases}$$

In particular for $n = 3$, $\beta = 0$ is

$$(2)\qquad |a_5| \leqslant \tfrac{1}{2} + e^{-2/3}.$$

The inequalities (1) and (2) are generalizations of some previous results [1].

Using (1) and a sharp estimation proved by **M. M. Schiffer**, see [2], I shall prove that $|a_5| \leqslant 5$ in a certain subclass $\bar{S}$ of S defined below, the equality holds only for Köbe function (which belongs to $\bar{S}$).

Let us first remind some formulas given by **F. Leja** [5]. A domain D which is the image of $|z| < 1$ under the transformation $f(z) \in S$ contains the point $w = 0$. If $\zeta = 1/w$, then $D \to \Delta$ where Δ contains $\zeta = \infty$. The complementary set to Δ is a bounded continuum E with capacity $d(E) = 1$. The point $\zeta = 0$ belongs to E. To every function $f(z) \in S$ corresponds a continuum E, $d(E) = 1$ and a coordinate system with the origin $0 \in E$. If E and 0 are fixed the rotation of the coordinate system transfers the function $f(z) \in S$ into $f(ze^{i\alpha}) = e^{i\alpha}z + a_2 e^{2i\alpha}z^2 + a_3 e^{3i\alpha}z^3 + \ldots$ and therefore we obtain a new function $g(z) = \dfrac{f(ze^{i\alpha})}{e^{i\alpha}} = z + a_2 e^{i\alpha}z^2 + \ldots$, $g(z) \in S$. The absolute value of the coefficients of $g(z)$ are the same as those of $f(z)$. The translation of the origin $0 \in E$ gives a new function $\in S$.

Let μ be the natural mass-distribution of the unit mass on E and let $\bar{0}$ be the center of gravity of this distribution. The point $\bar{0}$ belongs to E or not. For example if E denotes an arch of a circle, then $\bar{0} \notin E$. When E is a segment of the lenght equal to 4, then $\bar{0}$ is the middle of E. We define a subclass $\bar{S}$ of S in the following way:

$f(z) \in \bar{S}$ if and only if the corresponding continuum E contains the point $\bar{0}$.

In [5] the following formulas were proved

$$a_2 = -s_1, \quad a_3 = \frac{3s_1^2 - s_2}{2}, \dots \tag{3}$$

where

$$s_k = \int_E \zeta^k d\mu(\zeta), \quad k = 1, 2, 3, \dots$$

In particular $0\tilde{0} = s_1 = \int_E \zeta d\mu(\zeta)$. Put $\zeta = s_1 + \xi$, then

$$s_k = \int_E (s_1 + \xi)^k d\mu = s_k(\bar{0}) + \binom{k}{1} s_{k-1}(\bar{0}) s_1 + \binom{k}{2} s_{k-2}(\bar{0}) s_1^2 + \dots + s_1^k . \tag{4}$$

2. **Theorem.** *For* $f(z) \in \bar{S}$ *is* $|a_5| \leqslant 5$.

Proof. Using (3) and (4) we obtain

$$a_5 = \left[-\frac{s_4(\bar{0})}{4} + \frac{3}{4} s_2^2(\bar{0}) \right] + \left[2a_2 a_4 - \frac{a_2^4}{2} - \frac{a_3^2}{2} \right]. \tag{5}$$

M. M. Schiffer proved [2] that

$$\left| 2a_2 a_4 - \frac{a_3^2}{2} - \frac{a_2^4}{2} \right| \leqslant \frac{7}{2} \tag{6}$$

for all $f(z) \in S$, the equality holds only for Köbe function.

From (1) follows ($n = 3$, $a_2 = 0$) for $\beta \leqslant 1 - \frac{3}{2} = -\frac{1}{2}$

$$|a_5 - \beta a_3^2| = \left| -\frac{s_4(\bar{0})}{4} + \left(\frac{5}{8} - \frac{\beta}{4} \right) s_2^2(\bar{0}) \right| \leqslant 1 - \beta .$$

Let be $\alpha = \frac{5}{8} - \frac{\beta}{4}$, then for $f(z) \in \bar{S}$

$$\left| -\frac{s_4(\bar{0})}{4} + \alpha s_2^2(\bar{0}) \right| \leqslant 4\alpha - \frac{3}{2}, \quad \alpha \geqslant \frac{3}{4}, \tag{7}$$

the equality holds for Köbe function.

Similarly

$$\left| -\frac{s_4(\bar{0})}{4} + \alpha s_2^2(\bar{0}) \right| \leqslant -4\alpha + \frac{3}{2} \quad \text{for} \quad \alpha \leqslant \frac{1}{4}, \ f(z) \in \bar{S} \tag{8}$$

and

$$(9)\qquad \left|-\frac{s_4(\bar{0})}{4}+a s_2^2(\bar{0})\right| \leqslant \frac{1}{2}+e^{\frac{-2(4a-3)}{1-4a}} \quad \text{for} \quad \frac{1}{4}<a\leqslant\frac{3}{4},\ f(z)\in\tilde{S}.$$

From (5), (6) and (7) follows for $a=3/4$

$$|a_5| \leqslant \frac{3}{2}+\frac{7}{2}=5, \qquad f(z)\in\tilde{S}.$$

Remark. I proved in [3] that the inequality (8) holds for all $f(z)\in S$. In [2] was proved the inequality $\left|-\frac{s_4(\bar{0})}{4}+\frac{3}{8}s_2^2(\bar{0})\right| \leqslant \frac{1}{2}+e^{-6}$ for all $f(z)\in S$ (i.e. the inequality (9) for $a=3/8$ holds for all $f(z)\in S$). If one proves that for $a=3/4$ (7) is true for all $f(z)\in S$, then $|a_5|$ should be $\leqslant 5$ for all $f(z)\in S$.

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