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Nr CXXXI PRACE MATEMATYCZNE. ZESZYT 11 1966

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On a theorem of M. Ghermănescu

1. One of the earlier papers of M. Ghermănescu [1] contains a result which may be formulated as follows:

Suppose that we are given a linear functional equation of the first order

$$\varphi[f(x)] = \lambda(x)\varphi(x) + \gamma(x)\,, \tag{1}$$

where $\lambda(x)$, $\gamma(x)$ and $f(x)$ denote known functions. For any three different solutions of equation (1) *the relation*

$$\frac{\varphi_3(x) - \varphi_2(x)}{\varphi_1(x) - \varphi_2(x)} = \alpha(x) \tag{2}$$

holds, where $\alpha(x)$ is an automorphic function, i.e.

$$\alpha[f(x)] = \alpha(x)\,. \tag{3}$$

This relation characterizes the linear functional equations of the first order, which have the property that their general solution may be expressed in terms of two distinct particular solutions and an arbitrary automorphic function:

$$\varphi(x) = \alpha(x)\varphi_1(x) + [1-\alpha(x)]\varphi_2(x)\,. \tag{4}$$

The above theorem seems to be of a particular interest. It gives the general solution of equation (1) in a very simple form, whenever two particular solutions are known. The construction of all functions satisfying (3) is rather simple and may be realized without appealing to the axiom of choice (cf. [6]).

However, after a closer examination the situation turns out not so good as it seemed to be. The above theorem of Ghermănescu (which perhaps goes back as far as C. Popovici [7]; cf. also [2]) is not quite clear, an ambiguity being caused by the use of the words: *different*, *distinct*. If *different* is meant as *non-identical*, then the theorem is not true. To see this, consider the equation

$$\varphi(x+1) = \varphi(x) + 2x + 1\,, \tag{5}$$

with non-identical particular solutions $\varphi_1(x) = x^2$ and $\varphi_2(x) = x^2+\sin 2\pi x$. Then the family of functions given by (4) does not contain e.g. $\varphi(x) = x^2+1$, since setting $x = 0$ in (4) gives $\varphi(0) = 0$ for every φ.

If, on the other hand, *different* is meant as *different at every point*, then the theorem loses most of its interest. It is known (cf. e.g. [3], [4], [5], [6]) that in general equations of form (1) have a great number of non-identical solutions which coincide on quite large sets. More than that, there exist equations whose all solutions must coincide at some points. Such is e.g. the Schröder equation (cf. [6])

$$\varphi[f(x)] = \sigma\varphi(x)\,, \tag{6}$$

where $f(0) = 0$ and $\sigma \neq 1$. Every solution of equation (6) must vanish at $x = 0$. So such interpretation of the words *different, distinct* would exclude some very important particular cases of equation (1) from the area of applications of the above theorem.

The purpose of the present note is to give a precise formulation of the above result and to provide a rigorous proof.

2. In order to be able to prove anything we must state more precisely under what conditions equation (1) is considered. We shall assume that the independent variable ranges over a set E of a quite arbitrary nature. $f(x)$ is a function from E into E:

$$f(E)\subset E\,. \tag{7}$$

The functions $\lambda(x)$ and $\gamma(x)$ and the unknown function $\varphi(x)$ assume values in a number field Φ. In the sequel Latin letters will denote elements of E and Greek letters elements of Φ.

By *a solution of equation* (1) *in* E we shall understand a function $\varphi(x)$ defined in E, assuming values in Φ and satisfying (1) for every $x \in E$.

In the set E an equivalence relation ι may be defined (cf. [6]): $x_1 \iota x_2$ if and only if there exist non-negative integers n, m such that $f^n(x_1) = f^m(x_2)$. Here $f^i(x)$ denotes the i-th iterate of the function $f(x)$:

$$f^0(x) \equiv x\,, \quad f^{i+1}(x) = f[f^i(x)]\,, \quad i = 0, 1, 2, \ldots \tag{8}$$

If the function $f(x)$ is invertible, relation (8) defines the iterates $f^i(x)$ also for negative integers i.

The elements of the quotient space E/ι are called *cycles*. Thus

$$E = \bigcup C_j\,, \quad C_{j'} \cap C_{j''} = \emptyset \quad \text{for} \quad j' \neq j''\,.$$

For a given $x_0 \in E$ the cycle containing x_0 will be denoted by $C(x_0)$.

For a fixed integer k, $0 < k < \infty$, we shall denote by E_k the set of $x \in E$ with the following property: there exists a non-negative integer $i = i(x)$ such that $f^{k+i}(x) = f^i(x)$ and the relation $f^{l+j}(x) = f^j(x)$, $0 < l < k$, does not hold for any $j \geqslant 0$. The rest of E will be denoted by E_0,

$$E_0 = E - \bigcup_{k=1}^{\infty} E_k .$$

We have the following (cf. [6])

Lemma 1. *If $x_0 \in E_k$, $0 \leqslant k < \infty$, then $C(x_0) \subset E_k$.*

Next we define a decomposition of the set E into two parts:

$$E = A \cup B . \tag{9}$$

An $x_0 \in E$ belongs to A if and only if there exist two solutions $\varphi_1(x)$ and $\varphi_2(x)$ of equation (1) in E such that

$$\varphi_1(x_0) \neq \varphi_2(x_0) . \tag{10}$$

Otherwise $x_0 \in B$.

The decomposition (9) depends on equation (1) and has a non-effective character. A characterisation of the sets A, B can also be given in other terms.

Let us fix an $x_0 \in E$ and suppose that $\lambda(x) \neq 0$ for $x \in C(x_0)$. We define a set $V[x_0]$ as follows. If $x_0 \in E_0$, then we put $V[x_0] = \Phi$. If $x_0 \in E_k$, $k > 0$, then $V[x_0]$ is the set of those $\eta \in \Phi$ which fulfil the equation

$$\prod_{l=0}^{k+i-1} \lambda[f^l(x_0)]\eta + \sum_{l=0}^{k+i-2} \prod_{j=l+1}^{k+i-1} \lambda[f^j(x_0)]\gamma[f^l(x_0)] + \gamma[f^{k+i-1}(x_0)]$$
$$= \prod_{l=0}^{i-1} \lambda[f^l(x_0)]\eta + \sum_{l=0}^{i-2} \prod_{j=l+1}^{i-1} \lambda[f^j(x_0)]\gamma[f^l(x_0)] + \gamma[f^{i-1}(x_0)] ,$$

where $i = i(x_0)$ is the smallest integer $\geqslant 0$ for which $f^{k+i}(x_0) = f^i(x_0)$ (if $i = 0$, then the right-hand side of the above equation should be replaced simply by η).

Lemma 2. *Suppose that $\lambda(x) \neq 0$ for $x \in C(x_0)$ and that equation* (1) *has at least one solution in E. Then $x_0 \in A$ if and only if the set $V[x_0]$ contains at least two distinct elements.*

Proof. This follows directly from the results of our earlier paper [6] and from the fact that equation (1) can be considered independently on each cycle [1].

Lemma 3. *Suppose that $f(x)$ is invertible in E, $\lambda(x) = 0$ for some $x \in C(x_0)$ and equation* (1) *has at least one solution in E. Then $x_0 \in A$ if and only if $\lambda[f^{-i}(x_0)] \neq 0$ for all $i > 0$ such that $f^{-i}(x_0)$ is defined and belongs to E.*

[1] So if $\varphi_0(x)$ is a solution of equation (1) in E and $\varphi_1(x)$ is a solution of equation (1) in $C(x_0)$, then the function

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{for} \quad x \in C(x_0) \\ \varphi_0(x) & \text{for} \quad x \in E - C(x_0) \end{cases}$$

is a solution of equation (1) in E.

Proof. This can be deduced from the results stated in [5], but we shall provide an independent simple proof.

Suppose that there exists a $j > 0$ such that $f^{-j}(x_0) \in E$ and $\lambda[f^{-j}(x_0)] = 0$. Let $\varphi_1(x)$, $\varphi_2(x)$ be two arbitrary solutions of equations (1) in E (provided that they exist). Then we have

$$\varphi_1[f^{-j+1}(x_0)] = \lambda[f^{-j}(x_0)]\varphi_1[f^{-j}(x_0)] + \gamma[f^{-j}(x_0)] = \gamma[f^{-j}(x_0)]$$

$$\varphi_2[f^{-j+1}(x_0)] = \lambda[f^{-j}(x_0)]\varphi_2[f^{-j}(x_0)] + \gamma[f^{-j}(x_0)] = \gamma[f^{-j}(x_0)]$$

i.e. $\varphi_1[f^{-j+1}(x_0)] = \varphi_2[f^{-j+1}(x_0)]$. Induction then yields $\varphi_1[f^{-j+i}(x_0)] = \varphi_2[f^{-j+i}(x_0)]$, $i = 1, 2, 3, \ldots$ In particular for $i = j$ we get $\varphi_1(x_0) = \varphi_2(x_0)$, which shows that $x_0 \in B$.

Let us note that if, under the conditions of the lemma, $x_0 \in E_k$, $k > 0$, then $x_0 \in B$. In fact, suppose that for an $x' \in C(x_0)$ $\lambda(x') = 0$. In view of lemma 1 and of the invertibility of the function $f(x)$ we have

$$C(x_0) = C(x') = \{f(x'), \ldots, f^{k-1}(x'), f^k(x')\},$$

and consequently there exists a j, $1 \leqslant j \leqslant k$, such that $x_0 = f^j(x')$, i.e. $x' = f^{-j}(x_0)$ and $\lambda[f^{-j}(x_0)] = \lambda(x') = 0$. Thus $x_0 \in B$ on account of what has already been proved.

Now suppose that

(11) $$\lambda[f^{-i}(x_0)] \neq 0 \quad \text{for every} \quad i > 0$$

(whenever $f^{-i}(x_0) \in E$). We shall define two functions $\bar{\varphi}_1(x)$ and $\bar{\varphi}_2(x)$ on $C(x_0)$. Since $f(x)$ is invertible and since, in view of the preceding remark, $x_0 \in E_0$, we have

$$C(x_0) = \{\ldots f^{-2}(x_0), f^{-1}(x_0), x_0, f(x_0), f^2(x_0) \ldots\},$$

where by (7) the sequence $f^n(x_0)$ is infinite for $n > 0$ and infinite or finite for $n < 0$. We put

(12) $$\bar{\varphi}_1(x_0) = 0, \quad \bar{\varphi}_2(x_0) = 1$$

and then define $\bar{\varphi}_l(x)$, $l = 1, 2$, for $x \in C(x_0)$ recurrently:

(13)
$$\bar{\varphi}_l[f^{i+1}(x_0)] = \lambda[f^i(x_0)]\bar{\varphi}_l[f^i(x_0)] + \gamma[f^i(x_0)], \quad i = 0, 1, 2, \ldots$$
$$\bar{\varphi}_l[f^{i-1}(x_0)] = \frac{\bar{\varphi}_l[f^i(x_0)] - \gamma[f^{i-1}(x_0)]}{\lambda[f^{i-1}(x_0)]}, \quad i = 0, -1, -2, \ldots$$

In view of (11) the functions $\bar{\varphi}_l(x)$, $l = 1, 2$, are by (12) and (13) unambiguously defined in the whole of $C(x_0)$.

By hypothesis equation (1) has in E at least one solution $\varphi_0(x)$. The functions

$$\varphi_l(x) = \begin{cases} \bar{\varphi}_l(x) & \text{for} \quad x \in C(x_0), \\ \varphi_0(x) & \text{for} \quad x \in E - C(x_0), \end{cases} \qquad l = 1, 2,$$

satisfy equation (1) in E and by (12) fulfil condition (10). Thus $x_0 \in A$.

We end this section with one more lemma concerning the sets A and B.

Lemma 4. *If $\lambda(x) \neq 0$ in E, then $f(A) \subset A$ and $f(B) \subset B$.*

Proof. Suppose that an $x_0 \in A$ and let $\varphi_1(x)$ and $\varphi_2(x)$ be two solutions of equation (1) in E such that (10) holds. For an indirect proof let us suppose that $f(x_0) \in B$. Then $\varphi_1[f(x_0)] = \varphi_2[f(x_0)]$. But, in view of (1), this leads to

$$\lambda(x_0)\varphi_1(x_0) + \gamma(x_0) = \lambda(x_0)\varphi_2(x_0) + \gamma(x_0), \tag{14}$$

i.e. $\varphi_1(x_0) = \varphi_2(x_0)$, contrary to the supposition.

Now let us suppose that $x_0 \in B$. Then for any two solutions $\varphi_1(x)$ and $\varphi_2(x)$ of equation (1) in E (14) holds, which shows that $f(x_0)$ cannot belong to A. This completes the proof.

3. Now we shall prove the following

Theorem. *Suppose that the function $f(x)$ maps a set E into itself and the functions $\lambda(x)$ and $\gamma(x)$ are defined in E and assume values in a number field Φ. Further suppose that one of the following conditions is fulfilled:*

$$\lambda(x) \neq 0 \quad \textit{for} \quad x \in E, \tag{15}$$

$$f(x) \textit{ is invertible in } E. \tag{16}$$

Let $\varphi_1(x)$ and $\varphi_2(x)$ be two solutions of equation (1) in E such that

$$\varphi_1(x) \neq \varphi_2(x) \quad \textit{for} \quad x \in A. \tag{17}$$

Then the general solution of equation (1) in E is given by formula (4), where $a(x)$ is an arbitrary automorphic function with values in Φ.

Proof. It is easily seen that every function of form (4) with an a fulfilling (3), satisfies equation (1) in E. So we need only prove that for any solution $\varphi(x)$ of equation (1) in E there exists an automorphic function $a(x)$ such that relation (4) holds.

Let $\varphi(x)$ be a solution of equation (1) in E. We must distinguish two cases.

I. Suppose that condition (15) is fulfilled. Then we define the function $a(x)$ as follows:

$$a(x) = \begin{cases} \dfrac{\varphi(x) - \varphi_2(x)}{\varphi_1(x) - \varphi_2(x)} & \text{for} \quad x \in A, \\ 1 & \text{for} \quad x \in B. \end{cases} \tag{18}$$

According to (17) $a(x)$ is unambiguously defined in the whole of E. It follows from the fact that all the three functions $\varphi(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ satisfy equation (1) and from lemma 4 that $a(x)$ is an automorphic function. Now, if $x \in A$, then (4) results from (18). If, on the other hand, $x \in B$, then $\varphi(x) = \varphi_1(x) = \varphi_2(x)$ and (4) holds all the same.

II. Suppose that condition (16) is fulfilled. Then we define the function $\alpha(x)$ as follows:

$$\alpha(x) = \begin{cases} \dfrac{\varphi(x')-\varphi_2(x')}{\varphi_1(x')-\varphi_2(x')} & \text{if there exists an } x' \in A \cap C(x), \\ 1 & \text{otherwise.} \end{cases} \tag{19}$$

First we prove that $\alpha(x)$ is by (19) unambiguously defined in E.

In view of (17) the right-hand side of (19) has a meaning for every $x \in E$. To prove that $\alpha(x)$ does not depend on the choice of $x' \in A \cap C(x)$ suppose that also $x'' \in A \cap C(x)$, $x'' \neq x'$. Then $x'' \iota x'$, which means in view of (16) that there exists an integer n such that

$$x'' = f^n(x'). \tag{20}$$

Since $x' \neq x''$, $n \neq 0$. We may assume that $n > 0$. In view of lemma 3 we have

$$\lambda[f^j(x')] \neq 0 \quad \text{for} \quad j = 0, 1, \ldots, n-1. \tag{21}$$

Since $\varphi(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ satisfy equation (1) in E, we have by (21) for $j = 0, \ldots, n-1$

$$\begin{aligned} &\frac{\varphi[f^{j+1}(x')]-\varphi_2[f^{j+1}(x')]}{\varphi_1[f^{j+1}(x')]-\varphi_2[f^{j+1}(x')]} \\ &\quad = \frac{\{\lambda[f^j(x')]\varphi[f^j(x')]+\gamma[f^j(x')]\}-\{\lambda[f^j(x')]\varphi_2[f^j(x')]+\gamma[f^j(x')]\}}{\{\lambda[f^j(x')]\varphi_1[f^j(x')]+\gamma[f^j(x')]\}-\{\lambda[f^j(x')]\varphi_2[f^j(x')]+\gamma[f^j(x')]\}} \\ &\quad = \frac{\varphi[f^j(x')]-\varphi_2[f^j(x')]}{\varphi_1[f^j(x')]-\varphi_2[f^j(x')]}, \end{aligned}$$

whence we obtain

$$\frac{\varphi[f^{j+1}(x')]-\varphi_2[f^{j+1}(x')]}{\varphi_1[f^{j+1}(x')]-\varphi_2[f^{j+1}(x')]} = \frac{\varphi(x')-\varphi_2(x')}{\varphi_1(x')-\varphi_2(x')} \quad \text{for} \quad j = 0, 1, \ldots, n-1,$$

and in particular for $j = n-1$ (cf. (20))

$$\frac{\varphi(x'')-\varphi_2(x'')}{\varphi_1(x'')-\varphi_2(x'')} = \frac{\varphi(x')-\varphi_2(x')}{\varphi_1(x')-\varphi_2(x')}.$$

This shows that $\alpha(x)$ is unambiguously defined in E.

$\alpha(x)$ is an automorphic function, since $C(x) = C[f(x)]$ for every $x \in E$. We shall show that (4) is fulfilled for every $x \in E$.

If $x \in A$, then, as has been shown above, we may take in (19) $x' = x$, whence (4) follows immediately. If, on the other hand, $x \in B$, then $\varphi(x) = \varphi_1(x) = \varphi_2(x)$ and (4) holds independently of the value of $\alpha(x)$.

This completes the proof.

Remark. It follows from the results in [5] and [6] that if equation (1) has at least one solution in E, then there exist solutions $\varphi_1(x)$ and $\varphi_2(x)$ fulfilling (17).

4. The result of the present paper can also be applied in the case where the set E can be decomposed into two disjoint parts

$$E = E' \cup E'', \quad E' \cap E'' = \emptyset$$

such that $f(E') \subset E'$, $f(E'') \subset E''$ and on E' condition (15) and on E'' condition (16) is fulfilled. This results from the fact that (1) and (3) may be considered independently on each set E', E'', i.e. if φ', φ'' are solutions of (1) in E' and E'', respectively, and α', α'' are automorphic functions in E' and E'', respectively, then

$$\varphi = \begin{cases} \varphi' & \text{in} \quad E' \\ \varphi'' & \text{in} \quad E'' \end{cases}$$

is a solution of (1) in E, and similarly

$$\alpha = \begin{cases} \alpha' & \text{in} \quad E' \\ \alpha'' & \text{in} \quad E'' \end{cases}$$

is an automorphic function in E.

But conditions (15), (16) cannot be altogether omitted. If we drop them, then the theorem becomes false. This may be seen from the following example.

Let E be the set of the pairs of integers $x = (n, m)$. We define the function $g(m)$ for integral m by the formula

$$g(m) = \begin{cases} m-1 & \text{for} \quad m > 0 \\ 0 & \text{for} \quad m = 0 \\ m+1 & \text{for} \quad m < 0 , \end{cases} \tag{22}$$

and we put

$$f(x) = f(n, m) = \big(n+1, g(m)\big) .$$

Relation (7) evidently holds. Moreover we have for $i \geqslant |m|$

$$f^i(x) = f^i(n, m) = (n+i, 0) .$$

Suppose that we are given two points $x_1 = (n_1, m_1)$ and $x_2 = (n_2, m_2)$ in E. We have for $i = \max(|n_1 - n_2| + |m_1|, |n_1 - n_2| + |m_2|)$ and $j = n_1 - n_2 + i$

$$f^i(x_1) = f^i(n_1, m_1) = (n_1 + i, 0) ,$$

$$f^j(x_2) = f^j(n_2, m_2) = (n_2 + n_1 - n_2 + i, 0) = (n_1 + i, 0) ,$$

which shows that $x_1 \,\iota\, x_2$. Thus the whole set E consists of a single cycle.