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On the convergence to zero of oscillating solutions of some partial differential equations

In the present paper we shall consider the question of the convergence to zero of oscillating solutions of the equations:

(1) $$z_{xy} = f(x, y, z, z_x, z_y)$$

(2) $$z_{xx} = f(x, y, z, z_x, z_y)$$

(3) $$z_{yy} = f(x, y, z, z_x, z_y)$$

when $\sqrt{x^2+y^2} \to \infty$.

The general idea of the present paper is analogous to the idea which was exposed in [3] for the ordinary differential equations. The other results concerning the similar problem for ordinary equations were given in [2].

We shall consider solutions of (1) (resp. (2) and (3)) defined in the whole plane (x, y).

1. Let $\{r_n\}$ be an increasing sequence of non-negative numbers such that $\lim_{n\to\infty} r_n = \infty$. We introduce the following notation

$$p_n = \{(x, y)\colon r_n \leqslant \sqrt{x^2+y^2} < r_{n+1}\} \qquad (n = 1, 2, \ldots).$$

By a solution of (1) ((2) or (3)) we mean the function $z(x, y)$ which has the continuous derivatives z_x, z_y, z_{xy} (resp. z_x, z_y, z_{xx}, or z_x, z_y, z_{yy}) and fulfils (1) (resp. (2) or (3)).

A solution $z(x, y)$ of (1) ((2) or (3)) is said to be $\{r_n\}$-*oscillating* if it is defined in the whole plane (x, y) and if there exists a sequence $\{c_n\}$ of curves defined by the equations $x = x_n(t)$, $y = y_n(t)$ $(n = 1, 2, \ldots)$ such that $c_n \subset p_n$, c_n contains the boundary of a connected set s_n such that $\{(x, y)\colon \sqrt{x^2+y^2} \leqslant r_n\} \subset s_n$, $z\big(x_n(t), y_n(t)\big) = 0$ identically with respect to t.

We shall denote by q_n the minimal connected set which contains the curves c_n and c_{n+1}.

A solution $z(x, y)$ of (1) is said to be *x-regular $\{r_n\}$-oscillating* (*y-regular $\{r_n\}$-oscillating*) if there exists a sequence $\{d_n\}$ of curves such that $d_n \subset q_n$, d_n contains the boundary of a set s_n^* such that $s_n \subset s_n^*$, $z_x(x, y) = 0$ for $(x, y) \in d_n$ ($z_y(x, y) = 0$ for $(x, y) \in d_n$).

2. Theorem 1. *Let $f(x, y, z, p, q)$ be defined for all x, y, z, p, q, let $z(x, y)$ be a x-regular $\{r_n\}$-oscillating (y-regular $\{r_n\}$-oscillating) solution of* (1), *let $\{r_n\}$ be an increasing sequence of non-negative numbers such that*

$$\sigma_n = \sqrt{r_{n+1}^2 - r_n^2} \to 0\,, \qquad r_n \to \infty$$

as $n \to \infty$; let further the function f fulfils the condition

(4) $$|f(x, y, z, p, q)| \leqslant M(x, y, z, p, q) \quad \text{for} \quad |p| \leqslant h \quad (\text{for } |q| \leqslant h)$$

where $M(x, y, z, p, q)$ is a function defined for all x, y, z, p, q such that if $|p| \leqslant h$ ($|q| \leqslant h$) then for each sequence $\{(\xi_n, \eta_n)\}$, $(\xi_n, \eta_n) \in p_n$ $(n = 1, 2, \ldots)$, the following condition is satisfied

(5) $$\sigma_n \cdot M(\xi_n, \eta_n, z, p, q) \to 0 \quad \text{as} \quad n \to \infty$$

uniformly with respect to (z, p, q).

Under these assumptions

(6) $$\lim_{x^2+y^2\to\infty} z(x, y) = 0\,.$$

Proof. At first we shall show that for some N, if $n > N$, then

(7) $$|z_x(x, y)| \leqslant h \quad (|z_y(x, y)| \leqslant h) \quad \text{for} \quad (x, y) \in q_n\,.$$

Suppose that the inequality (7) does not hold for an infinite number of q_n. In each such a q_n there exists a point $(\hat{x}_n, \hat{y}_n)$ such that $|z_x(\hat{x}_n, \hat{y}_n)| > h$ ($|z_y(\hat{x}_n, \hat{y}_n)| > h$).

In virtue of the fact that $z(x, y)$ is x-regular oscillating (y-regular oscillating), for each fixed $x(y)$ and for each q_n, there exists $y_n(x_n)$ such that $(x, y_n) \in q_n$ and $z_x(x, y_n) = 0$ (resp. $(x_n, y) \in q_n$ and $z_y(x_n, y) = 0$). Hence for $\hat{x}_n$ there exist a_n and b_n (resp. for $\hat{y}_n$ there exist a_n and β_n) such that

$$|z_x(\hat{x}_n, y)| \leqslant h \quad \text{for} \quad y \in \langle a_n, b_n \rangle$$

and

$$z_x(\hat{x}_n, a_n) = 0\,, \quad z_x(\hat{x}_n, b_n) = h \quad \text{or} \quad z_x(\hat{x}_n, a_n) = h\,, \quad z_x(\hat{x}_n, b_n) = 0$$

respectively

(8) $$|z_y(x, \hat{y}_n)| \leqslant h \quad \text{for} \quad x \in \langle a_n, \beta_n \rangle$$

and

(9) $$z_y(a_n, \hat{y}_n) = 0\,, \quad z_y(\beta_n, \hat{y}_n) = h \quad \text{or} \quad z_y(a_n, \hat{y}_n) = h\,, \quad z_y(\beta_n, \hat{y}_n) = 0$$

Hence

$$h = |z_x(\hat{x}_n, a_n) - z_x(\hat{x}_n, b_n)| \leqslant |a_n - b_n| \max_{\langle a_n, b_n\rangle} |z_{xy}(\hat{x}_n, y)|$$

$$\leqslant |a_n - b_n| \cdot M\big(\hat{x}_n, \eta_n, z(\hat{x}_n, \eta_n), z_x(\hat{x}_n, \eta_n), z_y(\hat{x}_n, \eta_n)\big)$$

where η_n is a point of $\langle a_n, b_n\rangle$.

Hence

$$h \leqslant \sigma_n \cdot M\big(\hat{x}_n, \eta_n, z(\hat{x}_n, \eta_n), z_x(\hat{x}_n, \eta_n), z_y(\hat{x}_n, \eta_n)\big)$$

which contradicts to (5).

Similarly, by using of (8) and (9) (where z is y-regular oscillating) we obtain

$$h \leqslant \sigma_n \cdot M\big(\xi_n, \hat{y}_n, z(\xi_n, \hat{y}_n), z_x(\xi_n, \hat{y}_n), z_y(\xi_n, \hat{y}_n)\big)$$

which is impossible in virtue of (5).

Hence for $n > N$ the inequality (7) holds.

It is easy to see that

$$\max_{q_n} |z(x, y)| \leqslant \max_{q_n} |z_x(x, y)| \cdot \sigma_n, \qquad \max_{q_n} |z(x, y)| \leqslant \max_{q_n} |z_y(x, y)| \cdot \sigma_n .$$

Hence

$$\max_{q_n} |z(x, y)| \leqslant h \cdot \sigma_n \tag{10}$$

and the relation (6) holds.

3. Theorem 2. *Let $z(x, y)$ be an $\{r_n\}$-oscillating solution of* (2) (*an $\{r_n\}$-oscillating solution of* (3)). *Let the inequality* (4) *holds for $|p| \leqslant h$ ($|q| \leqslant h$), where $M(x, y, z, p, q)$ and $\{r_n\}$ have the properties supposed in Theorem* 1. *Then* $\lim_{x^2+y^2\to\infty} z(x, y) = 0$.

Proof. In this case the inequality (7) for $n > N$ follows directly from the Rolle's theorem on the same way as in the proof of theorem of M. Łuczyński [3]. Hence in virtue of (10) the conclusion of Theorem 2 holds.

4. Remark. Points (x, y) such that $z(x, y) = 0$ are called by some authors the *node points* (cf. [1], especially for the solution of the elliptic equations); the curves c_n are the sets of the node points of oscillating solutions. Such curves are called in [1] *node lines.*

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