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On some binary relations and relations of three arguments

In the present paper we consider abstract spaces with some binary relations and some relations of three arguments. By a *relation of partial ordering* we mean a binary relation which is reflexive, transitive and antisymmetric (cf. [1]).

1. Let X be an abstract space and let $R \subset X \times X$.

In the present note we shall assume that the relation R is transitive. For the relation R we introduce the following relation S:

$$xSy \overset{\text{df}}{\Longleftrightarrow} xRy \quad \text{or} \quad x = y\,. \tag{1.1}$$

Moreover we define

$$E(R) = \{(x, y)\colon\ xRy \text{ and } yRx\}\,.$$

If A is a non-empty subset of X, then we define:

$$B(A, R) = \{y \in X\colon\ x \in A \Rightarrow xRy\} \tag{1.2}$$

$$C(A, R) = \{z \in B(A, R)\colon\ y \in B(A, R) \Rightarrow zSy\} \tag{1.3}$$

Remark 1. If z and $\hat{z}$ are arbitrary elements of $C(A, R)$, then $(z, \hat{z}) \subset E(S)$. If $z \neq \hat{z}$, then $(z, \hat{z}) \in E(R)$.

Remark 2. If the set $E(R)$ is empty, then the relation S is a relation of partial ordering. In this case $E(S) = \{(z, z)\colon z \in X\}$ and for each $A \subset X$ the set $C(A, R)$ is empty or has exactly one element ($\sup A$).

Theorem 1. *If the relation S defined by* (1.1) *is such that for each pair $x, y \in X$ the set $C(\{x, y\}, S)$ has exactly one element* (*which we denote by $c(x, y)$*), *then S is a relation of partial ordering.*

Proof. It is trivial to prove that S is reflexive and transitive. Let now xSy and ySx. It is clear that xSx and ySx and moreover for each $z \in B(\{x, y\}, S)$

we have xSz. Hence $x = c(x, y)$. On an analogous way we obtain $y = c(x, y)$. Hence $x = y$ and S is antisymmetric.

2. Theorem 2. *Suppose that* 1. $A \subset X$ *and* A *is non-empty,* 2. V *is a mapping of* X *into* X *such that* $V(A) \subset A$, 3. *if* xRy, *then* $V(x)RV(y)$, 4. *the set* $Q = \{x \in A\colon xRV(x)\}$ *is non-empty,* 5. *the set* $C(Q, R)$ *is non-empty.*

Then for each $z \in C(Q, R)$ *we have* $(z, V(z)) \in E(R)$ *or* $z = V(z)$ *(in other words* $C(Q, R) \times V(C(Q, R)) \subset E(S)$).

Proof. The idea of this proof is the same as that of the proofs of theorems concerning partially ordered sets and increasing mappings in [3], [4], [5].

First we shall show that $V(Q) \subset Q$. Let $x \in Q$; then $xRV(x)$. By assumption 3, we have $V(x)RV(V(x))$ and in the consequence $V(x) \in Q$. Now we shall prove that if $z \in C(Q, R)$, then

$$z \in Q \quad \text{or} \quad z = V(z)\,. \tag{2.1}$$

Indeed, suppose that $z \in C(Q, R)$. Then for each $x \in Q$ we have xRz. Then for each $x \in Q$ we have $V(x)RV(z)$ and, by transitivity, $xRV(z)$. From the definition of $C(Q, R)$ it follows that $zSV(z)$. Hence $z \in Q$ and (2.1) holds, or $z = V(z)$ and (2.1) holds too. On the other hand, if $V(z) \in Q$, then $V(z)Rz$ which finishes the proof.

Remark 3. Theorem 1 in [3] is a special case of the above theorem; it is easy to see that if we put $xRy \Leftrightarrow x \leqslant y$, then $C(Q, R) = \{\sup Q\}$ and $E(R) = E(S) = \{(z, z)\colon z \in X\}$.

Remark 4. If $E(R)$ is empty (for example $xRy \Leftrightarrow x < y$), then, under assumptions 1-4 of Theorem 2, $C(Q, R) \cap Q$ is empty, or if we suppose the conditions 1, 2, 4, and $C(Q, R) \cap Q \neq 0$, then assumption 3 could not hold. In this case one can formulate the following modification of Theorem 2.

Suppose that conditions 1, 2, 4, 5 *are satisfied and* 3'. *if* xRy, *then* $V(x)SV(y)$. *Then* $C(Q, R)$ *has the unique element* z. *Moreover* $z = V(z)$.

The above modification of Theorem 2 is equivalent to Theorem 1 in [3] (see Remark 2).

Another modification of Theorem 2 (independent of the fact that $E(R) \neq 0$ or $E(R) = 0$) is the following

Theorem 3. *Suppose that conditions* 1, 2, 3' *are satisfied and moreover:* 4'. $Q' = \{x \in A\colon xSV(x)\} \neq 0$, 5'. $C(Q', S) \neq 0$. *Then* $P = [Q' \times V(Q')] \cap E(S) \neq 0$, $C(Q', S) \subset P$ *and* $C(P, S) = C(Q', S)$.

3. In order to give an illustration of Theorem 2, consider a family of subsets of a space Y and the relation $ZRW \overset{\text{df}}{\Leftrightarrow} Z \setminus W \subset K$, where K is a given fixed subset of Y. Suppose that there is a mapping $f\colon Y \supset Z \to f(Z) \subset Y$ such that if $Z \setminus K \subset W \setminus K$, then $f(Z \setminus K) \subset f(W \setminus K)$. If we denote by Q the family of all sets Z such that $ZRf(Z)$, then $Q \neq 0$ and

$$C(Q, R) = \Big\{\bigcup_{Z} \{Z\colon Z \in Q\}\Big\}\,.$$

Because all assumptions of Theorem 2 are satisfied, the conclusion of it holds, which means that $U \setminus f(U) \subset K$ and simultaneously $f(U) \setminus U \subset K$, where by U we shortly denote $\bigcup_Z \{Z\colon Z \in Q\}$.

4. We proved in Section 2 that Theorems 2 and 3 are generalizations of Theorem 1 from [3]. But on the other hand, it is possible to reduce Theorem 3 to the results from [3]. In order to prove it, we introduce the following equivalence relation E,

$$xEy \overset{\text{df}}{\iff} (x, y) \in E(S)$$

The factor-space X/E is partially ordered,

$$[x] \leqslant [y] \overset{\text{df}}{\iff} xSy .$$

Moreover if we define $\bar{V}\colon X/E \to X/E$ in such a way that

$$\bar{V}([x]) = [V(x)],$$

then all assumptions of Theorem 1 from [3] are satisfied and in the consequence the conclusion of it holds, which means that the assertion of Theorem 3 from Section 2 of the present paper holds.

5. Consider the space X with the relations R, S (cf. (1.1)) and the mapping V such that $xRy \Rightarrow V(x) R V(y)$. We define

$$G(z) = \{x\colon zRx\}, \quad Q(z) = \{x \in G(z)\colon xRV(x)\} .$$

It is easy to see that if for some $y \in X$ is $yRV(y)$, then

(5.1) $$V\big(G(y)\big) \subset G(y) .$$

It is possible to prove (using 5(.1)) a following modification of Theorem 2: *If for above mapping V there exists $y \in X$ such that $yRV(y)$ and $C(Q(y), R) \neq 0$, then $\big(\hat{y}, V(\hat{y})\big) \in E(S)$ for each $\hat{y} \in C\big(Q(y), R\big)$.*

6. On an analogous way as (1.2) and (1.3) we can define

(6.1) $$B^*(A, R) = \{y\colon x \in A \Rightarrow yRx\} ,$$

(6.2) $$C^*(A, R) = \{z \in B^*(A, R)\colon y \in B^*(A, R) \Rightarrow ySz\} .$$

It is possible to formulate and prove, using (6.1) and (6.2), theorems analogous to Theorems 2 and 3, corresponding to Theorem 2 in [3].

7. Consider the space X and the relation $T \subset X \times X \times X$ such that

(7.1) $$\text{if } T(x, y, z) \quad \text{and} \quad T(z, z, x), \quad \text{then} \quad T(z, x, y) .$$

Theorem 4. *Assume that*: 1. $V_i\colon X \to X$ $(i = 1, 2)$, 2. *if* $T(x, y, z)$, *then* $T\big(V_2(x), V_2(y), V_2(z)\big)$, 3. $Q = \big\{x\colon T\big(x, V_1(x), V_2(x)\big)\big\} \neq 0$, 4. *if* $T\Big(V_2(x), V_2\big(V_1(x)\big),$

$V_2(V_2(x))$, then $T(V_2(x), V_1(V_2(x)), V_2(V_2(x)))$. Under these assumptions, if $z \in Q$ and z fulfils for each $x \in Q$ the condition $T(z, z, x)$, then

$$T(V_2(z), z, V_1(z)). \tag{7.2}$$

Proof. We shall first prove that

$$V_2(Q) \subset Q. \tag{7.3}$$

Indeed, let $x \in Q$ which means that $T(x, V_1(x), V_2(x))$. By assumption 2 we have

$$T(V_2(x), V_2(V_1(x)), V_2(V_2(x))). \tag{7.4}$$

From (7.4) and condition 4 follows that

$$T(V_2(x), V_1(V_2(x)), V_2(V_2(x))).$$

Hence (7.3) holds.

Let $z \in X$ satisfy the condition

$$z \in Q, \quad x \in Q \Rightarrow T(x, x, z).$$

From $z \in Q$ follows that $V_2(z) \in Q$ and in the consequence

$$T(V_2(z), V_2(z), z). \tag{7.5}$$

From (7.5) and (7.1) in virtue of the fact that z, being an element of Q, fulfils the relation $T(z, V_1(z), V_2(z))$, follows (7.2).

Corollary. *If we assume that*

$$\text{if} \quad T(x, y, z) \quad \text{and} \quad T(z, y, x), \quad \text{then} \quad T(y, z, x), \tag{7.6}$$

then as the conclusion of Theorem 4 we have $T(V_1(z), V_2(z), z)$.

Remark 5. If R is a transitive binary relation, then the relation

$$T(x, y, z) \overset{\text{df}}{\Longleftrightarrow} xRy \quad \text{and} \quad yRz$$

fulfils (7.1) and (7.6).

If R is a relation of partial ordering and there exists $z = \sup Q$, then z is the common solution of the equations $z = V_i(z)$ $(i = 1, 2)$ and moreover it is the supremum of the set of such common solutions.

8. As an application of Theorem 4 we give the following

Theorem 5. *Suppose that $X = \langle a, b\rangle$, φ and ψ are real functions such that $\varphi(X) \subset X$, $\psi(X) \subset X$, for each $x \in X$ there exists $\lim\limits_{y\to x-} \varphi(y)$ and is equal to $\varphi(x)$, $\psi(x)$ is an increasing function, and $\varphi[\psi(x)] \equiv \psi[\varphi(x)]$. Suppose that the set*

$$Q = \{x\colon\ x \leqslant \varphi(x) \leqslant \psi(x)\}$$

is non-empty. Then $\hat{x} = \sup Q$ *is the common solution of the equations*

$$x = \varphi(x), \quad x = \psi(x). \tag{8.1}$$

Moreover it is the supremum of the set of all common solutions of (8.1) *and the solution of the equation* $x = \varphi[\psi(x)]$.

Proof. It is easy to see that $\hat{x} \in Q$. Moreover if we put

$$T(x, y, z) \overset{\mathrm{df}}{\Longleftrightarrow} x \leqslant y \leqslant z,$$

then all assumptions of Theorem 4 are satisfied and in the consequence

$$T\big(\hat{x}, \varphi(\hat{x}), \psi(\hat{x})\big) \quad \text{and} \quad T\big(\psi(\hat{x}), \hat{x}, \varphi(\hat{x})\big).$$

Hence $\hat{x} = \varphi(\hat{x}) = \psi(\hat{x})$.

From the definition of $\hat{x}$ as the supremum of Q follows that $\hat{x}$ is the supremum of the set of all common solutions of (8.1).

9. Consider the space X. Let $P(X)$ be the family of subsets of X, let w be a fixed point of X. Let moreover

$$f\colon X \ni x \to f(x) \in X, \quad g\colon P(X) \ni E \to g(E) \in X.$$

We define

$$H_f = \{E \subset X\colon f(E) \subset E\},$$
$$H_g = \{E \in P(X)\colon F \subset E \Rightarrow g(F) \in E\},$$
$$D = \bigcap_A \{A\colon A \in H_g \cap H_f,\ w \in A\}.$$

For a relation $T \subset X \times X \times X$ we define

$$yT_1(\xi)z \Leftrightarrow T(\xi, y, z), \quad T_1(\xi) = \{(y, z)\colon T(\xi, y, z)\}$$
$$zT_2(\eta)x \Leftrightarrow T(x, \eta, z), \quad T_2(\eta) = \{(z, x)\colon T(x, \eta, z)\}$$
$$xT_3(\zeta)y \Leftrightarrow T(x, y, \zeta), \quad T_3(\zeta) = \{(x, y)\colon T(x, y, \zeta)\}.$$

Lemma. *If for each* $\varrho \in E$ $\big(E \in P(X)\big)$

$$T_i(\varrho) = E \times E \quad (i = 1, 2, 3),$$

then $T = E \times E \times E$.

From this lemma and from the theorem of W. Felscher (cf. [2], p. 162) it follows the following

Theorem 6. *If for each* $\varrho \in D$ *the following conditions are satisfied:*

I. $\{z\colon T(\varrho, z, w)\} = D$, *if* $(y, z) \in [D \times D] \cap E\big(T_1(\varrho)\big)$ *then* $T\big(\varrho, y, f(z)\big)$, *for each* $y \in D$: $\{z\colon T(\varrho, y, z)\} \in H_g$,

II. $\{z\colon T(w, \varrho, z)\} = D$, *if* $(z, x) \in [D \times D] \cap E\big(T_2(\varrho)\big)$ *then* $T\big(f(x), \varrho, z\big)$, *for each* $z \in D$: $\{x\colon T(x, \varrho, z)\} \in H_g$,

III. $\{x\colon T(x, w, \varrho)\} = D$, *if* $(x, y) \in [D \times D] \cap E\big(T_3(\varrho)\big)$ *then* $T\big(x, f(y), \varrho\big)$, *for each* $x \in D$: $\{y\colon T(x, y, \varrho)\} \in H_g$, *then* $T = D \times D \times D$.

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