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On the method of successive approximations for some operator equations with applications to partial differential hyperbolic equations

In the present paper we apply the method of the successive approximations to some operator equations. We give here the proof of a theorem published without proof in [4]. This theorem was applied in [4] and [5] to prove some theorems concerning the existence and the uniqueness of solutions of some initial, boundary and mixed problems for the equation

$$z_{xy} = f(x, y, z, z_x, z_y) \tag{0.1}$$

or systems of such equations.

All results given here are proved by using the general idea of T. Ważewski (cf. [6], [7]).

1. By R_m^Ω we denote the space of the systems $f = \{f^1, \ldots, f^m\}$ of m real functions defined in Ω.

Definition 1.1. If $f = \{f^1, \ldots, f^m\}$, $g = \{g^1, \ldots, g^m\}$, $f, g \in R_m^\Omega$, then $f \leqslant g \overset{\text{df}}{\Longleftrightarrow} f^i(t) \leqslant g^i(t)$ for each $t \in \Omega$ and each i. If L and M are two operators defined in $\Delta \subset R_m^\Omega$, such that $L(\Delta), M(\Delta) \subset \Delta$, then $L \leqslant M \overset{\text{df}}{\Longleftrightarrow} L(h) \leqslant M(h)$ for each $h \in \Delta$.

Definition 1.2. If $f = \{f^1, \ldots, f^m\} \in R_m^\Omega$, then $|f| \overset{\text{df}}{=} \{|f^1|, \ldots, |f^m|\}$ and

$$\frac{\partial f}{\partial \tau} \overset{\text{df}}{=} \left\{\frac{\partial f^1}{\partial \tau}, \ldots, \frac{\partial f^m}{\partial \tau}\right\}, \quad \int f d\mu \overset{\text{df}}{=} \left\{\int f^1 d\mu, \ldots, \int f^m d\mu\right\}.$$

Definition 1.3. Let $f_n \in R_m^\Omega$ $(n = 1, 2, \ldots)$. The sequence $\{f_n\}$ is said to be convergent (uniformly convergent) to $f \in R_m^\Omega$, if for each i the sequence $\{f_n^i\}$ is convergent (uniformly convergent). Then $f = \lim\limits_{n\to\infty} f_n = \{\lim\limits_{n\to\infty} f_n^1, \ldots, \lim\limits_{n\to\infty} f_n^1\}$. A system $f \in C^k$, if for each i $f^i \in C^k$.

Definition 1.4. Let T be a closed rectangle in the plane (x, y).

By $P_0(T, M, M_1, M_2, N_1, N_2)$ we denote the set of $z \in R_m^T$ such that (a) $|z| \leqslant M$, (b) $z \in C^1$, (c) $|z_x| \leqslant M_1$ and $|z_y| \leqslant M_2$, (d) $|z_x(x, y) - z_x(x, \bar{y})| \leqslant N_1|y-\bar{y}|$ and $|z_y(x, y) - z_y(\bar{x}, y)| \leqslant N_2|x-\bar{x}|$.

By $P_1(T, M_1, N_1)$ we denote the set of $v \in R_m^T$ such that (e) $|v| \leqslant M_1$, (f) v is continuous, (g) $|v(x, y) - v(x, \bar{y})| \leqslant N_1|y-\bar{y}|$.

By $P_2(T, M_2, N_2)$ we denote the set of $w \in R_m^T$ such that (h) $|w| \leqslant M_2$, (i) w is continuous, (j) $|w(x, y) - w(\bar{x}, y)| \leqslant N_2|x-\bar{x}|$.

By $P_j^+(T, \ldots)$ we denote the set of non-negative elements of $P_j(T, \ldots)$ $(j = 0, 1, 2)$.

By $Q_0(T, M, M_1, M_2)$ we denote the set of non-negative $z \in R_m^T$ fulfilling condition (a) and

$$\text{(k)} \qquad |z(x, y) - z(\bar{x}, \bar{y})| \leqslant M_1|x-\bar{x}| + M_2|y-\bar{y}| .$$

By $Q_1(T, M_1, N_1)$ we denote the set of non-negative $v \in R_m^T$ fulfilling conditions (e), (g) and upper semi-continuous with respect to x.

By $Q_2(T, M_2, N_2)$ we denote the set of non-negative $w \in R_m^T$ fulfilling conditions (h), (j) and upper semi-continuous with respect to y.

By $R_1(T, M_1, N_1)$ we denote the set of non-negative $v \in R_m^T$ fulfilling conditions (e), (g) and summable with respect to x.

By $R_2(T, M_2, N_2)$ we denote the set of non-negative $w \in R_m^T$ fulfilling conditions (h), (j) and summable with respect to y.

We shall very often instead of $P_0(T, M, \ldots)$, $Q_0(T, M, \ldots)$, shortly write $P_0, Q_0, \ldots$ etc.

$$P = P_0 \times P_1 \times P_2, \qquad Q = Q_0 \times Q_1 \times Q_2, \qquad R = Q_0 \times R_1 \times R_2,$$

$$P^+ = P_0^+ \times P_1^+ \times P_2^+$$

$$P_{(k)}(T, M, M_1, M_2, N_1, N_2) = P(T, kM, kM_1, kM_2, kN_1, kN_2) .$$

On the analogous way we define $Q_{(k)}$, $R_{(k)}$ and $P_{(k)}^+$.

The systems $[z^0, z^1, z^2]$, $[u^0, u^1, u^2]$, ... belonging to P, Q, R or P^+, will be denoted by capitals Z, U, ...

We shall consider systems of three operators $L = [L_0, L_1, L_2]$ (or $\Phi = [\Phi_0, \Phi_1, \Phi_2]$) such that L_j is defined in either P_j, P_j^+, Q_j and R_j and $L_j(P_j) \subset P_j$, $L_j(Q_j) \subset Q_j$ and $L_j(R_j) \subset R_j$; then $L(P) \subset P$ will mean that $L_j(P_j) \subset P_j$. If $L = [L_0, L_1, L_2]$, $M = [M_0, M_1, M_2]$, then $L \leqslant M \Leftrightarrow L_j \leqslant M_j$ $(j = 0, 1, 2)$.

2. **Lemma 2.1.** *If a sequence* $\{Z_n\}$ $(Z_n = [z_n^0, z_n^1, z_n^2])$ *fulfils the following conditions* $Z_n \in Q$, $Z_n \leqslant Z_{n-1}$ $(n = 2, 3, \ldots)$, *then there exists* $\lim_{n\to\infty} Z_n = Z$ *and* $Z \in Q$.

Proof. In virtue of Definition 1.3, the sequences of elements belonging to Q have the same properties as the sequences of one-dimensional functions. Hence $\{Z_n\}$, being decreasing and bounded, converges to some $Z = [z^0, z^1, z^2]$. z_n^0 fulfils the Lipschitz condition with respect to both variables with the constants M_1 and M_2 (for each n). Hence $\lim\limits_{n\to\infty} z_n^0 = z^0$ fulfils the same condition. Hence it is easy to see that $z^0 \in Q_0$. Similarly, it is easy to see that $\lim\limits_{n\to\infty} z_n^i = z^i \in Q_i$ $(i = 1, 2)$.

Lemma 2.2. *If a sequence* $\{Z_n\}$ *fulfils the following conditions:* $Z_n \in R$, $Z_n \leqslant Z_{n+1}$ $(n = 1, 2, \ldots)$, *then* $\{Z_n\}$ *converges to* $Z \in R$.

The proof of this Lemma is analogous to that of Lemma 2.1.

Lemma 2.3. *If a sequence* $\{Z_n\}$ *fulfils the assumptions of Lemma* 2.1 *and* $\lim\limits_{n\to\infty} Z_n = 0$, *then* $\{Z_n\}$ *converges uniformly to zero.*

Proof. This lemma follows directly from the following two propositions:

A sequence $\{\varphi_n\}$ *of functions defined in a bicompact space* E, *such that* $\varphi_n(E) \subset E$, $\varphi_n \geqslant 0$, φ_n *upper semi-continuous*, $\varphi_n \geqslant \varphi_{n+1}$, $\lim\limits_{n\to\infty} \varphi_n = 0$, *converges uniformly to zero.*

Proof. Let $\varepsilon > 0$. The sets $E_n = \{\tau \in E\colon \varphi_n(\tau) \geqslant \varepsilon\}$ are bicompact and $E_n \subset E_{n-1}$. If $E_n \neq 0$ for each n, then from the Cantor's theorem follows that there exists $\tau_0 \in E_n$ such that $\varphi_n(\tau_0) \geqslant \varepsilon$ $(n = 1, 2, \ldots)$ what contradicts to $\varphi_n \to 0$. Hence $E_N = 0$ for some N. Hence $\varphi_N(\tau) < \varepsilon$ and $\varphi_n(\tau) < \varepsilon$ for $n \geqslant N$ and $\tau \in E$ (cf. [1]).

If function of two variables (x, y) *defined in* $T = \langle a_1, a_2\rangle \times \langle b_1, b_2\rangle$ *fulfils the Lipschitz condition with respect to one variable and is upper semi-continuous with respect to the other, then it is an upper semi-continuous function of two variables* (cf. [1]).

3. **Lemma 3.1.** *If an operator* L *fulfils the following conditions:*

1. L *is defined in* $Q = Q(T, M, M_1, M_2, N_1, N_2)$, $T = \langle a_1, a_2\rangle \times \langle b_1, b_2\rangle$,
2. a) $L \geqslant 0$, b) $L(Q) \subset Q$,
3. *if* $Z \leqslant W$, *then* $L(Z) \leqslant L(W)$,
4. *if* $Z_n \to Z$, $Z_n \in Q$, $Z \in Q$, *then* $L(Z_n) \to L(Z)$,
5. *the unique solution of the equation* $Z = L(Z)$ *in the set* Q *is* $Z = 0$, *then for each* $^*Z \in Q$ *such that* $L(^*Z) \leqslant {}^*Z$ *the sequence*

$$Z_0 = {}^*Z, \quad Z_{n+1} = L(Z_n) \quad (n = 1, 2, \ldots) \tag{3.1}$$

converges uniformly to $Z \equiv 0$ *in* T.

Proof. The sequence (3.1) fulfils all assumptions of Lemma 2.1 and in the consequence there exists the limit $Z = \lim\limits_{n\to\infty} Z_n$ and $Z \in Q$. Hence, from the definition of $\{Z_n\}$ and assumption 4 it follows that $Z = L(Z)$. From assumption 5 it follows that $Z = 0$. Hence the sequence (3.1) fulfils all assumptions of Lemma 2.3 and in the consequence the convergence is uniform.

Lemma 3.1′. *Suppose that assumptions* 1-3 *and* 5 *of Lemma* 3.1 *are satisfied and assume moreover that:*

2. c) $L(P^+) \subset P^+$,

4′. *if* $Z_n \to Z$, $Z_n \in P^+$, $Z \in Q$, *then* $L(Z_n) \to L(Z)$. *Then for each* $^*Z \in P^+$ *such that* $L(^*Z) \leqslant {}^*Z$ *the sequence* (3.1) *converges uniformly in* T *to zero.*

Proof. In virtue of the inclusion $P^+ \subset Q$ the proof of Lemma 3.1′ is going by the same manner as that of Lemma 3.1.

Remark 3.1. It is possible to replace assumption 4 (or assumption 4′) by the following condition

4″. *if* $Z_n \searrow Z$, *then* $L(Z_n) \to L(Z)$.

From Lemma 3.1′ follows directly

Lemma 3.2. *Assume conditions* 1, 2a)-2c), 3 *and* 4 (*or* 4′ (4″)) *and moreover*

6. *if* $z \in P_0^+$ *and* $\bar{z} = L_0(z, z_x, z_y)$, *then* $\bar{z}_x = L_1(z, z_x, z_y)$ *and* $\bar{z}_y = L_2(z, z_x, z_y)$,
7. *the unique solution* $[z, p, q] \in Q$ *of the system*

$$z = L_0(z, p, q), \qquad p = L_1(z, p, q), \qquad q = L_2(z, p, q)$$

is $z = p = q = \hat{0}$.

Then for each $\tilde{z} \in P_0^+$ *such that*

$$L_0(\tilde{z}, \tilde{z}_x, \tilde{z}_y) \leqslant \tilde{z}, \qquad L_1(\tilde{z}, \tilde{z}_x, \tilde{z}_y) \leqslant \tilde{z}_x, \qquad L_2(\tilde{z}, \tilde{z}_x, \tilde{z}_y) \leqslant \tilde{z}_y$$

the sequences $\{z_n\}, \{p_n\}, \{q_n\}$ *defined by the formulas:*

$$z_0 = \tilde{z}, \qquad p_0 = \tilde{z}_x, \qquad q_0 = \tilde{z}_y$$

$$z_{n+1} = L_0(z_n, p_n, q_n), \quad p_{n+1} = \frac{\partial z_{n+1}}{\partial x}, \qquad q_{n+1} = \frac{\partial z_{n+1}}{\partial y} \qquad (n = 1, 2, \ldots)$$

are uniformly convergent in T *to zero.*

Definition 3.1. $\hat{Z}$ is said to be *the maximal solution* of the equation

$$Z = L(Z) \tag{3.2}$$

in the set A, if $\hat{Z}$ is a solution of (3.2), $\hat{Z} \in A$ and for each solution $W \in A$ of (3.2) is $W \leqslant \hat{Z}$.

Definition 3.2. Let L be defined in R and let $L(R) \subset R$. L fulfils *the condition* (W), if the maximal solution $\hat{Z}$ of (3.2) in R exists and $\hat{Z} \in Q$.

Lemma 3.3. *If* L *is defined in* R, $L(R) \subset R$ *and* L *fulfils assumptions* 1-4 *of Lemma* 3.1 (*or* 1-4′ *of Lemma* 3.1′), *then* L *fulfils the condition* (W).

Proof. From assumption 2 follows that for each $Z \in R(T, M, M_1, M_2, N_1, N_2)$ the following inequality

$$0 \leqslant L(Z) \leqslant \tilde{M} = [M, M_1, M_2]$$

holds. Evidently $\tilde{M} \in R$. Hence $L(\tilde{M}) \leqslant \tilde{M}$.

The existence of the maximal solution $\hat{Z}$ of (3.2) in R follows from Theorem 1 in [3] (or from the results of [2]). We shall show that $\hat{Z} \in Q$ proving by the way the existence of $\hat{Z}$.

Consider the sequence

$$Z_0 = \tilde{M}, \qquad Z_{n+1} = L(Z_n) \qquad (n = 0, 1, \ldots).$$

This sequence fulfils the assumptions of Lemma 2.1. Hence $\lim\limits_{n\to\infty} Z_n = \hat{Z} \in Q$. Moreover it is easy to see that $\hat{Z}$ is a solution of (3.2).

On the other hand, for each solution $W \in R$ of (3.2) we have $W \leqslant \widetilde{M}$. From this it follows that for each n the inequality $W \leqslant Z_n$ holds. Hence

$$W \leqslant \hat{Z} = \lim_{n\to\infty} Z_n . \tag{3.3}$$

From (3.3) follows that $\hat{Z}$ is the maximal solution of (3.2) in R. On the other hand $\hat{Z} \in Q$, which finishes the proof.

Lemma 3.4. *If L defined in $R = R(T, M, M_1, M_2, N_1, N_2)$ is increasing and fulfils the following conditions*

8. $L(R) \subset R$,
9. *if* $Z_n \to Z$, $Z_n \leqslant Z_{n+1}$, $Z_n \in R$, $Z \in R$, *then* $L(Z_n) \to L(Z)$, *and moreover* $W \in R$ *fulfils the inequality* $W \leqslant L(W)$, *then*

$$W \leqslant \hat{Z}, \tag{3.4}$$

where $\hat{Z}$ is the maximal solution of (3.2) *in R.*

Proof. This lemma is a corollary of Theorem A with Remark 4 in [2] (cf. too [3]). It is possible to prove it by using the following sequence

$$Z_0 = W, \quad Z_{n+1} = L(Z_n) \quad (n = 0, 1, \ldots)$$

which converges to a solution $Z \in R$ (cf. Lemma 2.2) of (3.2). It is easy to see that

$$Z_0 \leqslant Z_n \leqslant Z \leqslant \hat{Z} \quad (n = 1, 2, \ldots)$$

which finishes the proof.

Lemma 3.5. *Assume that L is defined in $R_{(3)}$, $L \geqslant 0$, L is increasing, L fulfils the following conditions*

10. $L(Q_{(3)}) \subset Q$, $\quad L(R_{(3)}) \subset R$,
11. *if* $Z_n \in Q_{(3)}$, $\quad Z_n \to Z \in Q_{(3)}$, $\quad$ *then* $\quad L(Z_n) \to L(Z)$,
    if $Z_n \in R_{(3)}$, $\quad Z_n \to Z \in R_{(3)}$, $\quad$ *then* $\quad L(Z_n) \to L(Z)$.

Let furthemore $U = [u^0, u^1, u^2] \in Q_{(2)}$. Then for each $W \in Q_{(3)}$ which fulfils the inequality

$$W \leqslant L(W) + U$$

the inequality

$$W \leqslant \hat{Z} \tag{3.5}$$

holds, where $\hat{Z}$ is the maximal solution of the equation

$$Z = L(Z) + U \tag{3.6}$$

in the set $Q_{(3)}$.

Proof. If we put $L^* = L + U$, then for L^* and $R_{(3)}$ are satisfied all assumptions of Lemma 3.4 corresponding to L and R. Hence (3.5) is the consequence of (3.4). From Lemma 3.3 follows that the maximal solution Z of (3.6) in $R_{(3)}$ is an element of $Q_{(3)}$.

4. **Theorem 1.** *Let us assume that*

1. $T = \{(x, y)\colon a_1 \leqslant x \leqslant a_2,\ b_1 \leqslant y \leqslant b_2\}$,
2. *the operator* $\Phi = [\Phi_0, \Phi_1, \Phi_2]$ *is defined in* $P = P(T, M, M_1, M_2, N_1, N_2)$ *and* $\Phi(P) \subset P$,
3. *if* $z \in P_0(T, M, M_1, M_2, N_1, N_2)$ *and* $\bar{z} = \Phi_0(z, z_x, z_y)$, *then* $\bar{z}_x = \Phi_1(z, z_x, z_y)$ *and* $\bar{z}_y = \Phi_2(z, z_x, z_y)$.
4. *if* $\{Z_n\}$ $(Z_n \in P,\ n = 1, 2, \ldots)$ *is uniformly convergent to* $Z \in P$, *then* $\{\Phi(Z_n)\}$ *is convergent to* $\Phi(Z)$.
5. $L = [L_0, L_1, L_2]$ *is defined in* $\widetilde{R}_{(3)} = R_{(3)}(T, \widetilde{M}, \widetilde{M}_1, \widetilde{M}_2, \widetilde{N}_1, \widetilde{N}_2)$ *where* $2M \leqslant 3\widetilde{M}$, $2M_i \leqslant 3\widetilde{M}_i$, $2N_i \leqslant 3\widetilde{N}_i$ $(i = 1, 2)$ *and*

(α) $L \geqslant 0$, $L(\widetilde{Q}_{(3)}) \subset \widetilde{Q}$, $L(\widetilde{R}_{(3)}) \subset \widetilde{R}$,

(β) $Z \leqslant W \Rightarrow L(Z) \leqslant L(W)$,

(γ) *if* $\{W_n\}$ $(W_n \in \widetilde{R}_{(3)},\ n = 1, 2, \ldots)$ *is convergent to* $W \in \widetilde{R}_{(3)}$, *then* $\{L(W_n)\}$ *is convergent to* $L(W)$,

(δ) *the unique solution of the equation* $Z = L(Z)$ *in the set* $Q_{(3)}(T, \widetilde{M}, \widetilde{M}_1, \widetilde{M}_2, \widetilde{N}_1, \widetilde{N}_2)$ *is* $Z = 0$,

(ε) *if* $Z, W \in P(T, M, M_1, M_2, N_1, N_2)$, *then*

$$|\Phi(Z) - \Phi(W)| \leqslant L(|W - Z|).$$

Then for each $^*Z \in P(T, M, M_1, M_2, N_1, N_2)$ *which has the form* $^*Z = [z, z_x, z_y]$, *such that there exists* $^*W \in Q_{(3)}(T, \widetilde{M}, \widetilde{M}_1, \widetilde{M}_2, \widetilde{N}_1, \widetilde{N}_2)$ *fulfilling the conditions* $L(^*W) \leqslant {}^*W$ *and*

$$|^*Z - \Phi(^*Z)| \leqslant {}^*W \tag{4.1}$$

the following sequence

$$Z_0 = {}^*Z, \qquad Z_{n+1} = \Phi(Z_n) \qquad (n = 1, 2, \ldots) \tag{4.2}$$

converges uniformly to the unique (in P*) solution of the equation*

$$Z = \Phi(Z). \tag{4.3}$$

Proof. I. Consider the sequence

$$W_0 = {}^*W, \qquad W_{n+1} = L(W_n) \qquad (n = 1, 2, \ldots). \tag{4.4}$$

We shall show by induction that for each n the inequality

$$|Z_{n+1} - Z_n| \leqslant W_n \tag{4.5}$$

holds. For $n = 0$ (4.5) follows directly from (4.1). Suppose (4.5) for $n = k-1$; hence

$$|Z_{k+1}-Z_k| = |\Phi(Z_k)-\Phi(Z_{k-1})| \leqslant L(|Z_k-Z_{k-1}|) \leqslant L(W_{k-1}) = W_k$$

which finishes the proof.

II. For $m \geqslant n$ we have

$$\begin{aligned}|Z_n-Z_m| &\leqslant |Z_n-Z_{n+1}|+|Z_m-Z_{m+1}|+|Z_{n+1}-Z_{m+1}| \\ &\leqslant W_n+W_m+L(|Z_n-Z_m|) \leqslant 2W_n+L(|Z_n-Z_m|)\end{aligned}$$

since $\{W_n\}$ is decreasing.

If we denote $S_{nm} = |Z_n-Z_m|$ and $U_n = 2W_n$, then

$$S_{nm} \leqslant L(S_{nm})+U_n.$$

It is easy to see that

$$S_{nm} \in Q_{(2)} = Q_{(2)}(T, M, M_1, M_2, N_1, N_2)\,. \tag{4.6}$$

From (4.6) and assumption 5 follows that

$$S_{nm} \in \widetilde{Q}_{(3)} = Q_{(3)}(T, \widetilde{M}, \widetilde{M}_1, \widetilde{M}_2, \widetilde{N}_1, \widetilde{N}_2)\,.$$

Now we can apply Lemma 3.5 and we have

$$S_{nm} \leqslant \hat{W}_n \tag{4.7}$$

where $\hat{W}_n$ is the maximal solution in $\widetilde{Q}_{(3)}$ of the equation $Z = L(Z)+U_n$.

From Lemma 3.1 follows that $\{U_n\}$ is convergent to zero as $n\to\infty$ and that the convergence is uniform. Moreover $U_{n+1} \leqslant U_n$ and in consequence $L(Z)+U_{n+1} \leqslant L(Z)+U_n$. If we denote by W_{n+1} the maximal solution of the equation $Z = L(Z)+U_{n+1}$ in the set $\widetilde{Q}_{(3)}$, then from Lemma 3.5 follows that $\hat{W}_{n+1} \leqslant \hat{W}_n$ $(n = 1, 2, \ldots)$.

From Lemma 2.1 follows that $\hat{W}_n \to W \in \widetilde{Q}_{(3)}$. Hence if we are going to the limits in the equality

$$\hat{W}_n = L(\hat{W}_n)+U_n\,,$$

then $W = L(W)$. From this and assumption 5 (δ) follows that

$$W = \lim \hat{W}_n = 0\,. \tag{4.8}$$

From Lemma 2.3 in virtue of (4.8) follows that the convergence is uniform. Hence from (4.7) follows that for $n\to\infty$, $m \geqslant n$, the sequence $\{S_{mn}\}$ is uniformly convergent to zero and in consequence we have the uniform convergence of $\{Z_n\}$ to some $Z \in P(T, M, M_1, M_2, N_1, N_2)$. From assumption 4 and the equalities $Z_{n+1} = \Phi(Z_n)$ $(n = 1, 2, \ldots)$ it follows that $Z = \Phi(Z)$. Hence Z is a solution of (4.3).