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Nr CXXXI PRACE MATEMATYCZNE. ZESZYT 11 1966

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Generalizations of a theorem of Fatou

1. Introduction. The purpose of this paper is to give two generalizations to several complex variables of the following classical theorem

Theorem of Fatou ([2], p. 199). *If*

$$\sum_{k=0}^{\infty} a_k \lambda^k$$

is a power series with positive radius of convergence r, then there exists a sequence $\{\varepsilon_k\}$, *where* $\varepsilon_k = 1$ *or* -1 *such that the function* $f(z)$ *given by*

$$f(z) = \sum_{k=0}^{\infty} \varepsilon_k a_k \lambda^k$$

cannot be holomorphically continued through any point of the circle $|\lambda| = r$.

Let

$$\sum_{\mu=0}^{\infty} a_\mu(z), \qquad a_\mu(z) = a_{\mu_1 \ldots \mu_n} z_1^{\mu_1} \ldots z_n^{\mu_n}, \tag{1}$$

be a multiple power series of n complex variables and let

$$\sum_{m=0}^{\infty} P_m(z), \qquad P_m(z) = \sum_{\mu_1+\ldots+\mu_n=m} a_\mu(z), \tag{2}$$

be a series of homogeneous polynomials of n complex variables. If $n=1$, then the both series (1) and (2) are simple power series with the center at 0. We shall prove the two following generalizations of Theorem of Fatou.

Theorem 1. *If the domain of convergence D of* (1) *is not empty, then there exists a multiple sequence* $\{\varepsilon_\mu\} = \{\varepsilon_{\mu_1 \ldots \mu_n}\}$, $\varepsilon_\mu = 1$ *or* -1, *such that the function* $g(z)$ *given by*

$$g(z) = \sum_{\mu=0}^{\infty} \varepsilon_\mu a_\mu(z), \qquad z \in D, \tag{3}$$

cannot be continued through any boundary point of D.

Theorem 2. *If the domain of convergence G of* (2) *is not empty, then there exists a sequence* $\{\varepsilon_m\}$, $\varepsilon_m = 1$ *or* -1 *such that the function* $h(z)$ *given by*

$$h(z) = \sum_{m=0}^{\infty} \varepsilon_m P_m(z), \quad z \in G, \tag{6}$$

cannot be continued through any boundary point of G.

The method of proofs, we present below, is a simple adaptation of the method applied in the case of $n = 1$. It is based on the following classical theorem on lacunary power series.

Theorem of Hadamard ([2], p. 198). *The sum of the series*

$$\sum_{k=1}^{\infty} a_{m_k} \lambda^{m_k},$$

where m_k *are positive integers such that* $m_{k+1} \geqslant (1+\theta)m_k$, θ *being a fixed positive number, cannot be continued through any point of its circle of convergence.*

By the way we get the following generalization of this theorem to several complex variables:

Let $\{m_k\}$ *be a sequence of positive integers such that* $m_{k+1} \geqslant (1+\theta)m_k$, θ *being a fixed positive number. Then the sum of the series*

$$\sum_{k=1}^{\infty} P_{m_k}(z), \tag{4}$$

where $P_{m_k}(z)$ *is a homogeneous polynomial of* n *complex variables of degree* m_k, *cannot be continued beyond the domain of convergence of* (4).

A subset E of the space C^n of n complex variables is called *n-circular* if

$$\mathring{z} \in E \Rightarrow \{z\colon\ |z_k| = |\mathring{z}_k|,\ k = 1, \ldots, n\} \subset E.$$

A subset F of C^n is called *circular* if

$$\mathring{z} \in F \Rightarrow \{z\colon\ z_k = \lambda\mathring{z}_k,\ k = 1, \ldots, n,\ |\lambda| = 1\} \subset F.$$

It is well known ([1], pp. 32-40) that if E is an n-circular domain containing 0, then any function holomorphic in E may be developed in a multiple power series of the form (1) convergent in E. Similarly, any function holomorphic in a circular domain F containing 0 may be developed in F in a series of homogeneous polynomials of the form (2). This along with Theorems 1 and 2 gives the following

Corollary. *An n-circular domain D (circular domain G) containing* 0 *is a domain of holomorphy if and only if it is a domain of convergence of a multiple power series of the form* (1) (*a series of homogeneous polynomials of the form* (2)).

The Corollary is a simple consequence of a general theorem of Cartan-Thullen ([1], p. 30). But it seems to be of some interest to obtain it at the introductory level of the theory of several complex variables, especially if one starts from the well developed theory of convergence of the series (1) and (2).

2. **Proof of Theorem 1.** Let $H = \{\zeta^{(k)} = (\zeta_1^{(k)}, \ldots, \zeta_n^{(k)}),\ k = 1, 2, \ldots\}$ be a countable subset of the boundary ∂D of D such that $\zeta_1^{(k)} \ldots \zeta_n^{(k)} \neq 0$, $k = 1, 2, \ldots$ and the closure of H is equal to ∂D. It is well known ([2], p. 512) that

$$\limsup_{|\mu|\to\infty} |a_\mu(z)|^{\frac{1}{|\mu|}} = 1\,, \qquad z \in H\,, \tag{5}$$

where $|\mu| = \mu_1 + \ldots + \mu_n$. Let N^n denote the set of all n-tuples of nonnegative integers $\mu = (\mu_1, \ldots, \mu_n)$. Let $\{\mu^{(k,l)}\}$, $k, 1 = 1, 2, \ldots$, be a subset of N^n such that $\lim\limits_{l\to\infty} |\mu^{(k,l)}| = \infty$, $k = 1, 2, \ldots$ and

$$\lim_{l\to\infty} |a_{\mu^{(k,l)}}(\zeta^{(k)})|^{\frac{1}{|\mu^{(k,l)}|}} = 1\,, \qquad k = 1, 2, \ldots \tag{6}$$

Let $\theta > 0$. Let $\{k_s\}$ be a sequence of positive integers such that each positive integer is repeated in $\{k_s\}$ infinitely many times (e.g. $\{k_s\} = \{1, 2, 1, 2, 3, 1, 2, 3, 4, 1, \ldots\}$). Given such a sequence $\{k_s\}$ define a strictly increasing sequence $\{l_s\}$ of positive integers such that the subsequence $\{\mu^{(s)}\}$ of $\{\mu^{(k,l)}\}$, where $\mu^{(s)} = \mu^{(k_s, l_s)}$ satisfies the inequalities

$$|\mu^{(s+1)}| \geqslant (1 + \theta)\, |\mu^{(s)}| \qquad s = 1, 2, \ldots \tag{7}$$

In view of (5) and (6) we have

$$\limsup_{s\to\infty} |a_{\mu^{(s)}}(z)|^{\frac{1}{|\mu^{(s)}|}} = 1\,, \qquad z \in H\,. \tag{8}$$

Therefore the multiple power series

$$\varphi(z) = \sum_{s=1}^{\infty} a_{\mu^{(s)}}(z) \tag{9}$$

(which is also a series of homogeneous polynomials of the form (2)) is convergent in D. We claim that the function φ cannot be continued through any point of ∂D (so D is a domain of holomorphy). Since H is dense in ∂D it is sufficient to prove that φ is not continuable through any point of H. Let $\mathring{z} \in H$. In virtue of (7), (8) and by Theorem of Hadamard the function

$$\Phi(\lambda) = \varphi(\lambda \mathring{z}) = \sum_{s=1}^{\infty} [a_{\mu^{(s)}}(\mathring{z})]\, \lambda^{|\mu^{(s)}|} \qquad |\lambda| < 1\,,$$

cannot be continued through the point $\mathring{\lambda} = 1$. So the function φ is not continuable through the point $\mathring{z}$.

The remaining part of the proof is set-theoretical and it is quite analogous as in the case of one complex variable. Namely, put $\varphi_0(z) = \sum\limits_{\mu=0}^{\infty} a_\mu(z) - \varphi(z)$ and write φ in the form

$$\varphi(z) = \varphi_1(z) + \varphi_2(z) + \ldots\,,$$

where each φ_k is a sum of unfinitely many terms of the series (9) and each $a_{\mu(s)}(z)$ is a term of exactly one φ_k. Let F denote the family of all series of the form

$$\varphi_0(z)+\varepsilon_1\varphi_1(z)+\varepsilon_2\varphi_2(z)+\ldots$$

where $\varepsilon_k=1$ or -1. The set F is nondenumerable and each its element g is of the form (3). If each series $g\in F$ were continuable through a point of H, there would exist two series g_1 and g_2 belonging to F and a point $\mathring{z}\in H$ such that g_1 and g_2 would be continuable through $\mathring{z}$. This would imply that g_1-g_2 is continuable through $\mathring{z}$. But $g_1(\lambda\mathring{z})-g_2(\lambda\mathring{z})$ is a lacunary power series of Hadamard. Therefore g_1-g_2 cannot be continued through the point $\mathring{z}$. The proof is completed.

3. Proof of Theorem 2. It follows from the elementary properties of domains of convergence of series of homogeneous polynomials ([2], pp. 526-527) that there exists a countable set $H=\{\zeta^{(k)}\}$ contained in C^n-G such that ∂G is contained in the closure of H and

$$\limsup_{m\to\infty}\sqrt[m]{|P_m(z)|}=1\,,\qquad z\in H\,. \tag{10}$$

Take a double sequence of positive integers $\{m_{kl}\}$ such that $\lim\limits_{l\to\infty} m_{kl}=\infty$, $k=1,2,\ldots$, and

$$\lim_{l\to\infty}\sqrt[m_{kl}]{|P_{m_{kl}}(\zeta^{(k)})|}=1\qquad k=1,2,\ldots \tag{11}$$

Let $\{k_s\}$ be a sequence of positive integers such that each positive integer is repeated in $\{k_s\}$ infinitely many times. Let $\{l_s\}$ be a strictly increasing sequence of positive integers such that the sequence $m_s=m_{k_s l_s}$, $s=1,2,\ldots$, satisfies the inequalities

$$m_{s+1}\geqslant(1+\theta)\,m_s\,,\qquad s=1,2,\ldots$$

In virtue of (10) and (11) we have

$$\limsup_{s\to\infty}\sqrt[m_s]{|P_{m_s}(z)|}=1\,,\qquad z\in H\,.$$

The function $\varphi(z)=\sum\limits_{s=1}^{\infty}P_{m_s}(z)$, $z\in G$, is not continuable through any point of ∂G. This is a simple consequence of Theorem of Hadamard by considering the function $\varphi(\lambda\mathring{z})=\sum\limits_{s=1}^{\infty}P_{m_s}(\mathring{z})\,\lambda^{m_s}$ where $\mathring{z}$ is an arbitrary fixed point of ∂G. The remaining part of the proof is set-theoretical and in principle it does not differ from the corresponding part of the proof of Theorem 1.

REFERENCES

[1] Б. А. Фукс, *Специальные главы теории аналитических функций многих комплексных переменных*, Москва 1963.

[2] F. Leja, *Teoria funkcji analitycznych*, Warszawa 1957.