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On the Lie derivative of geometric objects from the point of view of functional equations

In the present paper we state and solve the following problem: to give an adequate definition of a Lie derivative of a purely differential geometric object and to find an explicit expression for it. Our definition is closely related to that proposed by J. Aczél and S. Gołąb in [1]. The Lie derivative of a geometric object of the first class with respect to a vector field is defined there as some differential concomitant. We give here an extension to higher classes of objects and we imput some additional conditions which assure the uniqueness of solution. On the other hand our result is consistent with L. Ye. Yevtushik's result [2] obtained by E. Cartan's method.

A purely differential object of class p is submitted to a full differential pseudogroup $D^{p,n}$ of order p over an n-dimensional manifold B of class C^{∞}. The structure of this pseudogroup may be described as follows: every differentiable transformation of local coordinates $\{\xi^{\alpha}\}\to\{\xi^{\bar{\alpha}}\}$ on B involves the parameters

$$A^{\bar{\alpha}}_{\alpha_1\ldots\alpha_s}=\frac{\partial^s\xi^{\bar{\alpha}}}{\partial\xi^{\alpha_1}\ldots\partial\xi^{\alpha_s}}$$

which have the following composition rule

$$A^{\bar{\bar{\alpha}}}_{\alpha_1\ldots\alpha_m}=\sum_{s=1}^{m}A^{\bar{\bar{\alpha}}}_{\bar{\beta}_1\ldots\bar{\beta}_s}\sum_{K_s^m}\prod_{t=1}^{s}A^{\bar{\beta}_t}_{\alpha_{t_1}\ldots\alpha_{t_r}}$$

where K_s^m denotes the set of all groups of indices $\{(\alpha_{t_l})\}$ such that $\{\alpha_{t_l}\}$ is a permutation of $\{\alpha_l\}$ and the indices in parentheses satisfy inequalities $\alpha_{t_1}\leqslant\ldots\leqslant\alpha_{t_s}$. We associate also with every transformation $\xi^{\alpha}\to\xi^{\alpha}$ a field of parameters

$$C^{\alpha}_{\alpha_1,\ldots,\alpha_m,\sigma}=\sum_{s=1}^{m}A^{\alpha}_{\bar{\gamma}_1\ldots\bar{\gamma}_s}\sum_{K_s^m}\partial_\sigma\prod_{t=1}^{s}A^{\bar{\gamma}_t}_{\alpha_{t_1}\ldots\alpha_{t_r}}\quad\left(\partial_\sigma=\frac{\partial}{\partial\xi^{\sigma}}\right)\tag{1}$$

which appear in expressions of the transformation rule of the derivatives of a field of geometric objects.

We consider a field of geometric object Ω of the class C^{p+1} which is submitted to the following transformation rule

$$\Omega^{\bar{K}} = \Phi^{\bar{K}}(\Omega, A)$$

where A is a brief denotation of an element of $D^{p,n}$. We shall use the following denotations

$$\text{(2)} \quad \begin{cases} \Omega_H^{\bar{K}} = \dfrac{\partial \Phi^{\bar{K}}(\Omega, A)}{\partial \Omega^H}, \\ \hat{\Omega}^K|_a^{a_1 \ldots a_s} = \dfrac{\partial \Phi^K(\Omega, A)}{\partial A^{\bar{a}}_{a_1 \ldots a_s}}\Bigg|_{A^{\bar{a}}_\beta = \delta^a_\beta,\ A^a_{\beta_1\beta_2 \ldots} = 0}. \end{cases}$$

Suppose that simultaneously with Ω there is given a vector field V.

Definition. A differential concomitant $\mathfrak{L}_V\Omega$ will be called a *Lie derivative of Ω with respect to V*:

1°.

$$\mathfrak{L}_V\Omega^K = V^a\partial_a\Omega^K + \Psi^K(\Omega, \partial V)$$

where Ψ^K are differentiable functions, $\partial V = \{\partial_{a_1 \ldots a_s}V^a\}$ $s \leqslant p$.

2°. $\mathfrak{L}_V\Omega$ is provided with a following transformation rule: if $\{\xi^a\} \to \{\xi^{\bar{a}}\}$, $\Omega \to \bar{\Omega}$, *then*

$$\overline{\mathfrak{L}_V\Omega^K} = \Omega_H^{\bar{K}}\mathfrak{L}_V\Omega^H.$$

3°. $\mathfrak{L}_0\Omega^K = 0$.

Theorem. *The axioms of the above definition determine $\mathfrak{L}_V\Omega$ uniquely and we have*

$$\mathfrak{L}_V\Omega^K = V^a\partial_a\Omega^K - \sum_{s=1}^{p} \hat{\Omega}^K|_a^{a_1 \ldots a_s}\partial_{a_1 \ldots a_s}V^a.$$

Proof. To begin with we refer that the sequence $\partial_a\Omega^K$ is submitted to the following transformation rule

$$\text{(3)} \quad \overline{\partial_a\Omega^K} = A^\beta_{\bar{a}}\Omega_H^{\bar{K}}\Big[\partial_\beta\Omega^H + \sum_{t=1}^{p} \hat{\Omega}^H|_a^{a_1 \ldots a_t} C^a_{a_1 \ldots a_t, \beta}\Big]$$

where $\hat{\Omega}^H|_a^{a_1 \ldots}$ and $C^a_{a_1 \ldots}$ are defined above (1) and (2). This follows from a more general formula for objects submitted to any Lie group with a composition function $\theta(u; v)$. In such a general case we have

$$\overline{d\Omega^K} = \Omega_H^{\bar{K}}[d\Omega^H + \hat{\Omega}_s^H\theta^s(\overset{*}{v}, dv)]$$

(cf. [3]). In our case, when we consider A instead of v, $C^a_{a_1 \ldots}$ take place of $\theta^s(\overset{*}{v}, dv)$.

We shall prove our theorem by induction. For $p=1$, i.e. for the first class objects, conditions 1° and 2° of the definition and formula (3) imply that

$$(4)\qquad \Omega^{\overline{K}}_{H}[V^{\alpha}\partial_{\alpha}\Omega^{H}+\Psi^{H}(\Omega,\partial V)] = \Omega^{\overline{K}}_{H}[V^{\alpha}\partial_{\alpha}\Omega^{H}+\hat{\Omega}^{H}|^{\nu}_{\mu}C^{\mu}_{\nu\beta}V^{\beta}]+ \\ +\Psi^{K}(\Omega,\{A^{\bar{\alpha}}_{\varkappa\beta}V^{\beta}+A^{\bar{\alpha}}_{\beta}\partial_{\varkappa}V^{\beta}\})$$

where

$$C^{\mu}_{\nu\beta}=A^{\mu}_{\bar{\pi}}A^{\bar{\pi}}_{\nu\beta}.$$

Differentiating both members of (4) with respect to some parameter $A^{\bar{\alpha}}_{\varkappa\beta}$ and substituting

$$A^{\bar{\mu}}_{\nu}=\delta^{\mu}_{\nu},\qquad A^{\bar{\mu}}_{\lambda\nu}=0\qquad(\mu,\nu,\lambda=1,\dots,n)$$

we obtain

$$\frac{\partial\Psi^{K}}{\partial(\partial_{\beta}V^{\alpha})}+\hat{\Omega}^{K}|^{\beta}_{\alpha}=0.$$

We see that Ψ are linear in ∂V and, in virtue of axiom 3°, we have

$$\Psi^{K}(\Omega,\partial V)=-\hat{\Omega}^{K}|^{\mu}_{\nu}\partial_{\mu}V^{\nu}.$$

Thus we obtain $\mathfrak{L}_{V}\Omega$ in the form

$$\mathfrak{L}_{V}\Omega^{K}=V^{\alpha}\partial_{\alpha}\Omega^{K}-\hat{\Omega}^{K}|^{\mu}_{\nu}\partial_{\mu}V^{\nu}.$$

We may examine directly that this concomitant satisfies conditions 1° and 2° of the definition. Hence it follows that if Ω is a linear object, then $\mathfrak{L}_{V}\Omega$ is a geometric object; in the nonlinear case we must adjoin the components Ω^{L} to $\mathfrak{L}_{V}\Omega^{K}$ in order to obtain a complete geometric object.

Suppose now that our theorem is true for $p=1,\dots,k-1$. We shall prove it for $p=k$. We shall write arguments in Ψ more explicitely, namely

$$\Psi^{K}=\Psi^{K}(\Omega,\partial^{1}V,\dots,\partial^{p}V)$$

where $\partial^{r}V$ denotes the sequence $\{\partial_{\alpha_1\dots\alpha_r}V^{\alpha}\}$. In a first step we restrict our considerations to the pseudosubgroup $D^{k,n}_{k}$ of the transformations of the form

$$A^{\bar{\alpha}}_{\beta}=\delta^{\alpha}_{\beta},\qquad A^{\bar{\alpha}}_{\alpha_1\dots\alpha_r}=0\qquad\text{if}\qquad r=2,\dots,k-1.$$

If we apply any transformation from $D^{k,n}_{k}$ to the functions

$$V^{\alpha}\partial_{\alpha}\Omega^{K}+\Psi^{K}(\Omega,\partial^{1}V,\dots,\partial^{k-1}V,\partial^{k}V)$$

and if we impose condition 2° of the theorem, then we obtain the following equations

$$\Omega^{\overline{K}}_{H}[V^{\alpha}\partial_{\alpha}\Omega^{H}+\Psi^{H}(\Omega,\partial^{1}V,\dots,\partial^{k-1}V,\partial^{k}V)] \\ =\Omega^{\overline{K}}_{H}[V^{\alpha}\partial_{\alpha}\Omega^{H}+\hat{\Omega}^{H}|^{\alpha_1\dots\alpha_p}_{\alpha}A^{\bar{\alpha}}_{\alpha_1\dots\alpha_k,\beta}V^{\beta}]+ \\ +\Psi^{H}(\overline{\Omega},\partial^{1}V,\dots,\partial^{k-1}V,\{\partial_{\alpha_1\dots\alpha_k}V^{\alpha}+A^{\bar{\alpha}}_{\alpha_1\dots\alpha_k,\beta}V^{\beta}\}).$$

We differentiate both members with respect to $A^{\bar{a}}_{\beta_1 \dots \beta_k}$ and we substitute $A^{\mu}_{\nu_1 \dots \nu_k} = 0$. This yields

$$(5) \qquad \Psi^H(\Omega, \partial^1 V, \dots, \partial^{k-1} V, \partial^k V) = -\Omega^H|_a^{a_1 \dots a_k} \partial_{a_1 \dots a_k} V^k + \widetilde{\Psi}(\Omega, \partial^1 V, \dots, \partial^{k-1} V).$$

We apply to both members of this equality a transformation of the pseudosubgroup $D^{k-1,n}$. If we impose once more condition 2°, then we obtain the following equation for $\widetilde{\Psi}$:

$$\Omega^{\overline{K}}_H [V^a \partial_a \Omega^H + \widetilde{\Psi}^H(\Omega, \partial^1 V, \dots, \partial^{k-1} V)$$
$$= \Omega^{\overline{K}}_H \Big[V^a \partial_a \Omega^H + \sum_{s=1}^{k-1} \hat{\Omega}^H|_a^{a_1 \dots a_s} C^a_{a_1 \dots a_s, \beta} V^\beta \Big] + \widetilde{\Psi}^H(\overline{\partial^1 V}, \dots, \overline{\partial^{k-1} V}, \overline{\Omega}).$$

We observe that it is the same equation which is implied by the conditions of the definition for the objects of class $k-1$. Then we may apply our assumption of induction and we have a unique solution

$$\widetilde{\Psi}(\Omega, \partial^1 V, \dots, \partial^{k-1} V) = -\sum_{s=1}^{k-1} \hat{\Omega}^K|_a^{a_1 \dots a_s} \partial_{a_1 \dots a_s} V^a.$$

This equality adjoined to (5) yields us

$$(6) \qquad \mathfrak{L}_V \Omega^K = V^a \partial_a \Omega^K - \sum_{s=1}^{k} \hat{\Omega}^K|_a^{a_1 \dots a_s} \partial_{a_1 \dots a_s} V^a.$$

Since every element of $D^{k,n}$ may be represented in a unique manner as a composition of some element of $D^{k,n}_k$ and some element of $D^{k-1,n}$ and since all steps of our construction give unique solutions, (6) gives a unique object $\mathfrak{L}_V \Omega^K$ which satisfies the conditions of our definition.

Remark. The condition 3° means that if all derivatives ∂V vanish, then $\mathfrak{L}_V \Omega$ reduces to $V^a \partial_a \Omega$, as in the case of a scalar. It may be proved that condition 3° may be left out in the case of nonlinear object, but this condition is essential for the unicity in the case of linear objects.

REFERENCES

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