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On Absolute G_δ in Perfectly Normal Topological Spaces

A topological space is called *perfectly normal* if it is normal and if every closed set in it is of type G_δ. Let Y be a topological space and $X \subset Y$. The collection of sets of the form $A \cup B$, where A is open in Y and $B \subset Y \setminus X$, defines, as it is easy to see, a new topology on Y. This topological space will be denoted by $Y(X)$.

Let now Q be a collection of topological spaces. A set X is said to be an *absolute G_δ with respect to Q* if for every $Y \in Q$ and every homeomorphism h of X into Y $h(X)$ is G_δ in Y. B. H. McCandless [1] has proved that a topological space is an absolute G_δ with respect to the class of all metric spaces if and only if $Y\big(h(X)\big)$ is perfectly normal for every metric space Y and every homeomorphism h of X into Y. In the present paper we state an analogous result for the class of all perfectly normal spaces.

At first we shall prove the following:

Lemma. *Let Y be a perfectly normal space and $X \subset Y$. X is G_δ in Y if and only if $Y(X)$ is perfectly normal.*

Proof. If $Y(X)$ is a perfectly normal space, then, because X is closed in $Y(X)$, there exists a sequence $\{C_n\} = \{A_n \cup B_n\}$, where A_n is open in Y and $B_n \subset Y \setminus X$, such that $X = \bigcap_{n=1}^{\infty} C_n$. It is easy to see that $X \subset A_n$, for every n, so that $X = \bigcap_{n=1}^{\infty} A_n$. It follows from this that X is G_δ in Y. Let now X be G_δ in Y. To show that $Y(X)$ is perfectly normal, it suffices to show that for every closed set F in $Y(X)$ there exists a continuous function $f: Y(X) \to [0,1]$ such that $f^{-1}(0) = F$ (see [2]). Let F be a closed set in $Y(X)$. The set F is of the form $C \cap D$, where C is closed in Y and $D \supset X$. Let $X = \bigcap_{n=1}^{\infty} A_n$, A_n open in Y. We may assume that $A_n \supset A_{n+1}$. Let $f: Y \to [0,1]$ be a continuous function such that $f^{-1}(0) = C$. Putting

$$g(x) = \begin{cases} f(x) \text{ for } x \in (Y \setminus C) \cup F \\ 1/n \text{ for } x \in (A_n \setminus A_{n+1}) \cap (C \setminus F) \end{cases}$$

we get a continuous function of $Y(X)$ into $[0, 1]$ such that $g^{-1}(0) = F$. This completes the proof of the lemma.

Theorem. *A topological space X is an absolute G_δ with respect to the class Q of all perfectly normal spaces if and only if for every space $Y \in Q$ and every homeomorphism h of X into Y $Y(h(X))$ is a perfectly normal space.*

The proof of the theorem simply follows from the lemma.

REFERENCES

[1] B. H. McCandless, *Perfect normality, local neighbourhood extension spaces and contractibility of absolute retracts*, Math. Zeitschr. 85 (1964), 385-391.

[2] N. Vadenisoff, *Sur les fonctions continues dans les espaces topologiques*, Fund. Math. 27, (1936), 234-238.