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On Commutative Algebraic Objects over a Groupoid

The algebraic objects are considered by A. Zajtz in [2]. The purpose of this paper is discussing properties of special algebraic objects e.g., commutative algebraic objects. At first we show some theorems about commutative algebraic objects over a groupoid. Next we give an example of two similar, but non-equivalent algebraic objects. Theorem 6 gives sufficient conditions for equivalence of two similar algebraic objects.

Algebraic objects can be considered as a special kind of quasi-algebras (see in [1]). I agree with the author of [2] that a *non-singular algebraic object* I shall call a *complete algebraic object*. Instead of a homomorphism satisfying the condition (W) I shall say a *strong homomorphism* (according to the terminology for quasi-algebras, see in [1]). All terms, which I do not define in this paper, are used in the sense of definitions given in [2].

1. Let (A, X) be any algebraic object over X . For any subset U of A we denote by $O(U)$ an algebraic subobject generated by U . About (A, X) we always assume that for every $a \in A$, there exists at least one $x \in X$ such that a product xa is defined. Under this hypothesis, if U generates (A, X) then for each $a \in A$, there exist $u \in U$ and $x \in X$ such that $a = xu$.

Let (A, X) and (B, X) be two algebraic objects over X , and let $h: A \rightarrow B$ be a strong homomorphism of (A, X) into (B, X) . If (C, X) is an algebraic subobject of (A, X) then $(h(C), X)$ is a subobject of (B, X) . (It is worth to remark that above theorem is not true for any arbitrary (non-strong) homomorphism.)

An algebraic object (A, X) over X is called commutative if for any $a \in A$ and $x, y \in X$, if $x(ya)$ is defined then $y(xa)$ is also defined, and $x(ya) = y(xa)$.

At first we prove the following

Lemma 1. *Let x be such element of X that xa is defined for all $a \in A$. If an algebraic object (A, X) is commutative then the mapping $h_x(a) = xa$ is a strong homomorphism of (A, X) into itself.*

Proof. Observe that h_x is defined at every $a \in A$. If a product ya is defined then $x(ya)$ is defined too. Thus all expressions

$$(1) \quad h_x(ya) = x(ya) = y(xa) = yh_x(a)$$

are defined. h_x is a strong homomorphism, because if $yh_x(a)$ is defined then all expressions (1) are defined.

Now we prove the following theorem:

Theorem 1. *Let (A, X) be a complete algebraic object. (A, X) is commutative if and only if h_x is a homomorphism for every $x \in X$.*

Proof. The necessity follows from Lemma 1. If (A, X) is complete then all products are defined, and for all $x, y \in X$ and $a \in A$,

$$x(ya) = h_x(ya) = yh_x(a) = y(xa).$$

Before we formulate the next theorem we prove the following

Lemma 2. *Let an algebraic object (A, X) be transitive and commutative*. If $x \in X$ is such that a product xa is defined for some $a \in A$ then for all $b \in A$, a product xb is defined.*

Proof. For some $y \in X$, $a = yb$. Since all expressions $xa = x(yb) = y(xb)$ are defined, thus xb is defined.

Theorem 2. *If an algebraic object (A, X) is transitive and commutative then every homomorphism $h: A \rightarrow A$ of (A, X) into itself is equal h_x for some $x \in X$.*

Proof. Let $a_0 \in A$. There exists $x \in X$ such that $h(a_0) = xa_0$. By Lemma 2, xa is defined for every $a \in A$, thus by Lemma 1, h_x is defined, and $h_x(a_0) = xa_0 = h(a_0)$. Since a_0 generates (A, X) thus by (1.4) in [1] (page 7), $h = h_x$.

Corollary 1. *If an algebraic object (A, X) is transitive and commutative then*
 (a) *every homomorphism $h: A \rightarrow A$ of (A, X) into itself is a strong epimorphism*
 (b) *for every $a, b \in A$, there exists a strong epimorphism $h: A \rightarrow A$ such that $h(a) = b$.*

Proof. (a) By Theorem 2 and Lemma 1, $h = h_x$ is a strong homomorphism, thus $(h(A), X)$ is a subobject of (A, X) . Since (A, X) is transitive, $h(A) = A$.

(b) There exists $x \in X$ such that $xa = b$. By Lemma 2 and Lemma 1, h_x is a homomorphism, $h_x(a) = b$. By (a), h_x is a strong epimorphism.

2. Now we shall consider an algebraic object (A, G) over a groupoid G . Observe that, if a product xa is defined (for $a \in A$, $x \in G$) then $x^{-1}(xa)$ is defined, and $x^{-1}(xa) = \delta_x a = a$ (see in [2]). From this, we immediately obtain the following

Corollary 2. *If (A, G) is an algebraic object over a groupoid G , and $xa = xb$ for some $x \in G$, then $a = b$, because*

$$a = x^{-1}(xa) = x^{-1}(xb) = b.$$

We also remark the following:

Theorem 3. *If (A, G) is a simple algebraic object over a groupoid G then (A, G) is transitive.*

Proof. Let (A, G) be generated by a_0 , and let $a \in A$. Of course $O(a) \subset A$. Since $a = xa_0$ for some $x \in G$, thus $a_0 = x^{-1}a \in O(a)$. Hence $A = O(a_0) \subset O(a)$, and every $a \in A$ generates (A, X) .

* In [2] such an object is called *particular*.

Next we prove the following

Theorem 4. *If an algebraic object (A, G) over a groupoid G is simple and commutative, then*

- (a) *every homomorphism $h: A \rightarrow A$ of (A, G) into itself is a strong isomorphism*
- (b) *for every $a, b \in A$, there exists a strong isomorphism $h: A \rightarrow A$ such that $h(a) = b$.*

The proof of Theorem 4 follows immediately from Theorems 2, 3, Corollary 1 and the remark that for any $x \in G$, h_x is an injection.

Theorem 5. *If (A, G) is a commutative algebraic object over a groupoid G and U, V are two its generators, then there exists a strong isomorphism $h: A \rightarrow A$ of (A, G) onto (A, G) such that $h(U) = V$.*

At first we prove the following

Lemma 3. *If (A, G) is an algebraic object over a groupoid G and U is its generator then for every $a \in A$, there exists exactly one element $u \in U$ such that $a = xu$ (for some $x \in G$).*

Proof. If $a = x_1 u_1 = x_2 u_2$ for $u_1 \neq u_2$, $u_1, u_2 \in U$, $x_1, x_2 \in G$, then $u_1 = (x_1^{-1} x_2) u_2$. Thus $U - \{u_1\} \not\subseteq U$ generates (A, G) , which gives a contradiction.

Proof of Theorem 5. For $u \in U$ and $v \in V$, $O(u)$ and $O(v)$ are transitive (see Theorem 3). By Lemma 3, for any $u \in U$, there exists exactly one $v_u \in V$ such that $u = x_u v_u$ for some $x_u \in G$, thus the mapping $k: U \ni u \rightarrow v_u \in V$ is an injection. Since $k(U) \subset V$ generates (A, G) , $k(U) = V$ and k is a bijection. Since $u = x_u v_u$, thus $O(u)$ is a subobject of $O(v_u)$. Because $O(v_u)$ is transitive, $O(u) = O(v_u)$. By Theorem 4, there exists a strong isomorphism $h_u: O(u) \rightarrow O(v_u)$ such that $h_u(u) = v_u$. Let us put $h(a) = h_u(a)$, if $a \in O(u)$. h is well defined, because $O(u_1) \cap O(u_2) = \emptyset$ for $u_1 \neq u_2$. Since $k: U \rightarrow V$ is a bijection, thus $h: A \rightarrow A$ is a strong isomorphism of (A, G) onto (A, G) , and $h(U) = V$. The proof is complete.

3. Let (A, X) and (B, X) be two algebraic objects over X . (A, X) and (B, X) are said to be equivalent if there exists a strong isomorphism $h: A \rightarrow B$ of (A, X) onto (B, X) . (B, X) is a concomitant of (A, X) if there exists an epimorphism $h: A \rightarrow B$ of (A, X) onto (B, X) . (A, X) and (B, X) are called similar if one is a concomitant of other (see in [2]).

The following theorem gives the conditions under which every two similar algebraic objects are equivalent.

Theorem 6. *Let (A, G) and (B, G) be two commutative algebraic objects over a groupoid G , which have finite generators. If they are similar then they are equivalent.*

The proof of this theorem is based on the following:

Lemma 4. Let (A, G) and (B, G) be two transitive and commutative algebraic objects over a groupoid G . If there exist homomorphisms $h: A \rightarrow B$ and $g: B \rightarrow A$, then (A, G) and (B, G) are equivalent, as well as h and g are strong isomorphisms.

Proof. By Theorem 4, $gh: A \rightarrow A$ is a strong isomorphism. Let $h(a) = h(b)$, $a, b \in A$. Then $(gh)(a) = g(h(a)) = g(h(b)) = (gh)(b)$. Because gh is an injection, $a = b$. Thus h is an injection. If a product $xh(a)$ is defined then the expressions $g(xh(a)) = xg(h(a)) = x(gh)(a)$ are defined. Since gh is a strong isomorphism, a product xa is defined, and h is a strong monomorphism. Now, $(h(A), G)$ is a subobject of (B, G) . Because (B, G) is transitive, $h(A) = B$. Hence h is a strong isomorphism, and (A, G) , (B, G) are equivalent. In similar way, we prove that g is a strong isomorphism.

Proof of Theorem 6. Let $h: A \rightarrow B$ and $g: B \rightarrow A$ be epimorphisms. By the assumptions of Theorem 6, a generator of (A, G) and a generator of (B, G) have the same number n of elements. Now the proof is by the induction.

For $n = 1$, the theorem immediately follows from Lemma 4. Next, let (A, G) and (B, G) have generators with n elements, and let the theorem be true for $n - 1$. Consider a sequence $\{a_i\}$, $a_{i+1} = (gh)(a_i)$ ($i = 0, 1, 2, \dots$), where a_0 is an arbitrary (fixed) element of A . Since (A, G) has a finite generator, thus there exist numbers k and p such that $0 \leq k < p$ and $a_p = xa_k$ for some $x \in G$. Now *

$$(2) \quad a_p = (gh)(a_{p-1}) = (gh)^2(a_{p-2}) = \dots = (gh)^{p-k}(a_k).$$

We consider transitive algebraic objects $O(a_p)$ and $O(b_0)$, where $b_0 = h(a_p)$. Of course $h(O(a_p)) \subset O(b_0)$. Hence

$$f_1: O(a_p) \ni a \rightarrow h(a) \in O(b_0)$$

is a homomorphism. Using (2) we obtain

$$\begin{aligned} ((gh)^{p-k-1}g)(b_0) &= ((gh)^{p-k-1}g)(h(a_p)) = (gh)^{p-k}(a_p) = \\ &= (gh)^{p-k}(xa_k) = x(gh)^{p-k}(a_k) = xa_p \in O(a_p). \end{aligned}$$

Thus

$$f_2: O(b_0) \ni b \rightarrow ((gh)^{p-k-1}g)(b) \in O(a_p)$$

is a homomorphism. Now by Lemma 4, $O(a_p)$ and $O(b_0)$ are equivalent, and f_1, f_2 are strong isomorphisms. Hence $A - O(a_p)$ and $B - O(b_0)$ are similar (because $h(A - O(a_p)) = B - O(b_0)$ and $((gh)^{p-k-1}g)(B - O(b_0)) = A - O(a_p)$), and they have generators with $n - 1$ elements. By the induction hypothesis, they are equivalent. Thus (A, G) and (B, G) are equivalent.

4. The following example shows that Theorem 6 is not true for arbitrary (non-commutative) algebraic objects.

* If $\varphi: X \rightarrow X$ then $\varphi^0 = e_X$, and $\varphi^n = \varphi^{n-1}\varphi$.

Let Z be the set of integers, and let $I = \{-1, +1\}$. We construct two algebraic objects $(Z \times I, G, \cdot)$ and $(Z \times I, G, \circ)$, which are defined on the same set $Z \times I$, over some groupoid G , but the multiplications „ \cdot ” and „ \circ ” are different. The groupoid G and the operations „ \cdot ”, „ \circ ” we defined as follows. Denote by Z_1 the set of nonpositive integers and by Z_2 the union $Z_1 \cup \{2\}$. By T and S we denote the mappings

$$T: Z \times I \ni (n, \varepsilon) \rightarrow (n+1, \varepsilon) \in Z \times I$$

$$S: Z_2 \times I \ni (n, \varepsilon) \rightarrow (n, -\varepsilon) \in Z \times I$$

and by s the restriction of S to the set $Z_1 \times I$.

Let $A, B \subset C$, and let mappings $\varphi: A \rightarrow C$, $\psi: B \rightarrow C$ be such that $\varphi(A) \cap \psi(B) \neq \emptyset$. By $\psi\varphi$ we mean a mapping of $\varphi^{-1}(\varphi(A) \cap \psi(B))$ into C such that $(\psi\varphi)(x) = \psi(\varphi(x))$, $x \in \varphi^{-1}(\varphi(A) \cap \psi(B))$.

Let us introduce two families G_1, G_2 of mappings:

$$G_1 = \{f: f = f_1 \dots f_r, f_i = T \text{ or } T^{-1} \text{ or } S, i = 1, \dots, r; r \in N\}$$

$$G_2 = \{g: g = g_1 \dots g_r, g_i = T \text{ or } T^{-1} \text{ or } s, i = 1, \dots, r; r \in N\}$$

where N is the set of positive integers.

G_1 and G_2 are groupoids with the composition as the product. For $f = f_1 f_2 \dots f_r \in G_1$, where either $f_i = T$ or $f_i = T^{-1}$ or $f_i = S$, we put $F(f) = g = g_1 g_2 \dots g_r$, where

$$g_i = \begin{cases} f_i & \text{if } f_i = T \text{ or } f_i = T^{-1} \\ s & \text{if } f_i = S. \end{cases}$$

It is easy to see that the mapping $F: G_1 \rightarrow G_2$ is a strong isomorphism of G_1 onto G_2 (as a strong isomorphism of groupoids considered as quasi-algebras, see in [1]). Hence, there exists an universal model of G_1 and G_2 , e.g., there exist a groupoid G and strong isomorphisms $F_1: G \rightarrow G_1$, $F_2: G \rightarrow G_2$ such that a diagram

$$\begin{array}{ccc} & G & \\ F_1 \downarrow & \xrightarrow{\quad} & \downarrow F \\ G_1 & \xrightarrow{\quad F \quad} & G_2 \end{array}$$

is commutative. (It is sufficient to put $G = G_1$, $F_1 = e_{G_1}$, $F_2 = F$.)

Let $x \in G$, $a \in Z \times I$. We define the multiplications „ \cdot ”, „ \circ ” putting

$$x \cdot a = (F_1(x))(a), F_1(x) \in G_1$$

if the mapping $F_1(x)$ is defined at a ,

$$x \circ a = (F_2(x))(a), F_2(x) \in G_2$$

if the mapping $F_2(x)$ is defined at a .

$(Z \times I, G, \cdot)$ and $(Z \times I, G, \circ)$ are algebraic objects which are similar but not equivalent.

Firstly we shall show that they are not commutative. For this sake let us consider their diagrams:

the diagram of $(Z \times I, G, \cdot)$

$$\begin{array}{cccccccccccc} \longleftrightarrow & (-2, -1) & \xleftarrow{p^{-1}} & (-1, -1) & \xleftarrow{p^{-1}} & (0, -1) & \xleftarrow{p^{-1}} & (1, -1) & \xleftarrow{p^{-1}} & (2, -1) & \xleftarrow{p^{-1}} & (3, -1) & \longleftrightarrow \\ & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & \\ \longleftrightarrow & (-2, 1) & \xleftarrow{p^{-1}} & (-1, 1) & \xleftarrow{p^{-1}} & (0, 1) & \xleftarrow{p^{-1}} & (1, 1) & \xleftarrow{p^{-1}} & (2, 1) & \xleftarrow{p^{-1}} & (3, 1) & \longleftrightarrow \\ & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & \end{array}$$

the diagram of $(Z \times I, G, \circ)$

$$\begin{array}{cccccccccccc} \longleftrightarrow & (-2, -1) & \xleftarrow{p^{-1}} & (-1, -1) & \xleftarrow{p^{-1}} & (0, -1) & \xleftarrow{p^{-1}} & (1, -1) & \xleftarrow{p^{-1}} & (2, -1) & \xleftarrow{p^{-1}} & (3, -1) & \longleftrightarrow \\ & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & \\ \longleftrightarrow & (-2, 1) & \xleftarrow{p^{-1}} & (-1, 1) & \xleftarrow{p^{-1}} & (0, 1) & \xleftarrow{p^{-1}} & (1, 1) & \xleftarrow{p^{-1}} & (2, 1) & \xleftarrow{p^{-1}} & (3, 1) & \longleftrightarrow \\ & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & & \updownarrow q & \end{array}$$

Where p, p^{-1}, q are elements of G such that

$$(3) \quad \begin{cases} T = F_1(p) = F_2(p), & T^{-1} = F_1(p^{-1}) = F_2(p^{-1}) \\ S = F_1(q), & s = F_2(q) \end{cases}$$

and the symbol $(-1, -1) \xrightarrow{p} (0, 1)$ means that the product $p \cdot (-1, -1)$ [or $p \circ (-1, -1)$] is defined, and $p \cdot (-1, -1) = (0, -1)$ [or $p \circ (-1, -1) = (0, -1)$]. On these diagrams we mark out only p, p^{-1}, q but every $x = x_1 x_2 \dots x_r$, where either $x_i = p$ or $x_i = p^{-1}$ or $x_i = q$ ($i = 1, 2, \dots, r$).

It is easy to see that $(Z \times I, G, \cdot)$ and $(Z \times I, G, \circ)$ are generated by every $a \in Z \times I$, and they are not commutative because the products $p \cdot (q \cdot (0, -1))$ and $p \circ (q \circ (0, -1))$ are defined, while the products $q \cdot (p \cdot (0, -1))$ and $q \circ (p \circ (0, -1))$ are not defined (see the diagrams).

$(Z \times I, G, \cdot)$ and $(Z \times I, G, \circ)$ are similar. It follows from the fact that $h_1 = e_{Z \times I}$ is an epimorphism of $(Z \times I, G, \circ)$ onto $(Z \times I, G, \cdot)$ (but not inversely) and

$$h_2 : Z \times I \ni (n, \varepsilon) \rightarrow (n-2, \varepsilon) \in Z \times I$$

is an epimorphism of $(Z \times I, G, \cdot)$ onto $(Z \times I, G, \circ)$ (but not inversely). We shall show that h_1 is an epimorphism. Obviously, for all $a \in Z \times I$ and p, p^{-1} defined by (3), all products $p \circ a, p \cdot h_1(a), p^{-1} \circ a, p^{-1} \cdot h_1(a)$ are defined (see the diagrams), and we have

$$h_1(p \circ a) = p \cdot h_1(a), \quad h_1(p^{-1} \circ a) = p^{-1} \cdot h_1(a).$$

If $q \circ a$ is defined then $q \cdot h_1(a)$ is defined (because $h_1(Z_1 \times I) \subset Z_2 \times I$), as well as $h_1(q \circ a) = q \cdot h_1(a)$. Now, for every $x = x_1 x_2 \dots x_r \in G$, where either $x_i = p$ or $x_i = p^{-1}$ or $x_i = q$ ($i = 1, 2, \dots, r$), from above remarks it follows that

$$(4) \quad \begin{cases} h_1(x \circ a) = h_1(x_1 \circ (x_2 \dots x_r \circ a)) = x_1 \cdot h_1(x_2 \dots x_r \circ a) = \\ = x_1 x_2 \cdot h_1(x_3 \dots x_r \circ a) = x_1 x_2 \dots x_r \cdot h_1(a) = x \cdot h_1(a) \end{cases}$$

and, if some expression among (4) is defined then the next expression is defined too. Thus h_1 is an epimorphism of $(Z \times I, G, \circ)$ onto $(Z \times I, G, \cdot)$. Analogously, it can be shown that h_2 is an epimorphism of $(Z \times I, G, \cdot)$ onto $(Z \times I, G, \circ)$.

To prove that algebraic objects $(Z \times I, G, \cdot)$ and $(Z \times I, G, \circ)$ are not equivalent it is enough to show that there does not exist a strong monomorphism of $(Z \times I, G, \cdot)$ into $(Z \times I, G, \circ)$.

Assume contrary, let h be a strong monomorphism of $(Z \times I, G, \cdot)$ into $(Z \times I, G, \circ)$. Since the product $q \cdot (2, -1)$ is defined, thus the product $q \circ h(2, -1)$ must be defined. It is possible only in this case then $h(2, -1) = (n, \varepsilon)$, where $n \leq 0$, $\varepsilon \in I$. Now, because $(1, -1) = p^{-1} \cdot (2, -1)$, we obtain

$$h(1, -1) = h(p^{-1} \cdot (2, -1)) = p^{-1} \circ h(2, -1) = p^{-1} \circ (n, \varepsilon) = (n-1, \varepsilon).$$

The product $q \circ (n-1, \varepsilon) = q \circ h(1, -1)$ is defined (because $n-1 \leq 0$), while the product $q \cdot (1, -1)$ is not defined, and we get a contradiction.

REFERENCES

- [1] J. Słomiński, *A theory of extensions of quasi-algebras to algebras*, Rozprawy Matematyczne 40, Warszawa 1964.
 [2] A. Zajtz, *Algebraic objects*. Zeszyty Naukowe UJ, Prace Matematyczne, this issue, 67—79.