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A Note on the Uniqueness of Two Point Boundary Value Problems I

1. In the present note we are concerned with an ordinary differential equation of second order

$$x'' = f(t, x, x') \tag{1}$$

and the classical boundary value problem

$$\begin{aligned} \alpha x(a) + \beta x'(a) &= p \\ \gamma x(b) + \delta x'(b) &= q\,, \end{aligned} \tag{2}$$

where α, β, γ, δ, are real constants

If $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = 0$, then we obtain two point boundary value problem

$$x(a) = p\,, \quad x(b) = q\,. \tag{3}$$

If $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$, then (2) reduces to

$$x(a) = p\,, \quad x'(b) = q\,. \tag{4}$$

Already Ch. de la Vallée Poussin [2] observed that the uniqueness of problem (1), (4) implies the uniqueness of problem (1), (3). He used this fact in order to obtain a uniqueness criterion for problem (1), (3).

We prove here that under some assumptions also the uniqueness of problem (1), (2) implies the uniqueness of problem (1), (3). Using this fact and a result due to A. Lasota and Z. Opial we obtain a sufficient condition for the existence of a solution of problem (1), (3).

2. Let $f(t, x, y)$ be a function defined on $\Delta \times R \times R$, where Δ is an interval and R is real line.

Definition 1. Problem (1), (2) with any fixed $\alpha, \beta, \gamma, \delta$ is said to be globally unique if and only if for every $a, b \in \Delta$, $a \neq b$, and $p, q \in R$, there exists at most one solution of equation (1) satisfying (2) defined on Δ.

Definition 2. Problem (1), (2) with any fixed $\alpha, \beta, \gamma, \delta$ is said to be globally solvable if and only if for every $a, b \in \Delta$, $a \neq b$, and $p, q \in R$, there exists at least one solution of equation (1) defined on the whole Δ and such that (2) holds.

Condition C_0. We say that the function $f(t, x, y)$ satisfies condition C_0 if for every $a \in \Delta$ and $p, q \in R$ the initial Cauchy's problem

$$x(a) = p\,, \qquad x'(a) = q \tag{5}$$

for equation (1) admits at most one solution defined on Δ.

Condition C_1. We say that the function $f(t, x, y)$ satisfies condition C_1 if

1° $f(t, x, y)$ is continuous on $\Delta \times R \times R$, where Δ is an open interval.

2° for every $a \in \Delta$ and $p, q \in R$ there exists exactly one solution of problem (1), (5) and it is defined on the whole Δ.

Theorem 1. *Assume $f(t, x, y)$ satisfies condition C_0. If there exist $\alpha, \beta, \gamma, \delta$ such that*

$$\alpha\delta - \beta\gamma \neq 0 \tag{6}$$

and such that problem (1), (2) *is globally unique, then also problem* (1), (3) *is globally unique.*

Proof. Let $\beta, \delta \neq 0$ and let $x_1(t)$, $x_2(t)$ be solutions of problem (1), (3). Then for $x(t) = x_1(t) - x_2(t)$ we have

$$x(a) = 0\,, \qquad x(b) = 0\,. \tag{7}$$

Consider the functions

$$u(t) = \beta x(t) e^{\alpha t/\beta}\,, \qquad v(t) = \delta x(t) e^{\gamma t/\delta}\,. \tag{8}$$

From (7) and Rolle's theorem it follows that there exist $c, d \in (a, b)$ such that $u'(c) = 0$ and $v'(d) = 0$. Thus we have

$$\alpha x(c) + \beta x'(c) = 0\,,$$
$$\gamma x(d) + \delta x'(d) = 0\,.$$

If $c \neq d$, by the global uniqueness of (1), (2) we obtain $x(t) \equiv 0$. If $c = d$, by virtue of (6) we have $x(c) = x'(c) = 0$ and by condition C_0 $x(t) \equiv 0$ too.

In the case $\beta = 0$ ($\delta = 0$) we should consider instead of (8) the functions

$$u(t) = x(t)\,, \; v(t) = \delta x(t) e^{\gamma t/\delta} \qquad \big(u(t) = \beta x(t) e^{\alpha t/\beta},\; v(t) = x(t)\big)\,.$$

Remark 1. The converse theorem is not true. For the function $f(t, x, y) \equiv -x$ and for $\Delta = (0, \pi)$ problem (1), (3) is globally unique. But if

$$\cot(a - b) = \frac{\alpha\gamma + \beta\delta}{\alpha\delta - \beta\gamma}\,,$$

then we have the non-uniqueness of (1), (2).

Remark 2. Without the assumption (6) theorem 1 is not true. In fact, the equation

$$x''(t) = \frac{2t}{1+t^2} x' - \frac{2}{1+t^2} x$$

has the general solution of the form $x(t) = A(t^2 - 1) + Bt$ and consequently one and only solution satysfying the boundary value condition (3) with $\alpha = \beta = 0$,

$\gamma = \delta = 1$. But $x_1(t) \equiv 0$ and $x_2(t) = t^2-1$ are two different solutions of this equation such that

$$x_i(-1) = 0\,,\; x_i(1) = 0 \qquad i = 1, 2\,.$$

2. In a recent paper [1] A. Lasota and Z. Opial proved that the condition C_1 and the global uniqueness of (1), (2) (with $\beta = 0$) implies the global solvability of this problem. Hence by theorem 1 we obtain the following result:

Theorem 2. *Suppose the function* $f(t, x, y)$ *satisfies condition* C_1. *If* $\beta = 0$ *and* α, γ, δ *are such that problem* (1), (2) *is globally unique, then*

1° *problem* (1), (2) *with the same* $\alpha, \beta, \gamma, \delta$ *is globally solvable.*

2° *problem* (1), (3) *is globally unique and globally solvable.*

REFERENCES

[1] A. Lasota, Z. Opial, *On the existence and uniqueness of solutions of a boundary value problem for an ordinary second order differential equation.* Colloq. Math. 18 (1967), 7—11.

[2] Ch. de la Vallée Poussin, *Sur l'équation différentielle du second ordre,* Journ. Math. Pures et Appl. 8 (1929), 125-144.