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Application of the Differential Equations with Distributional Coefficients to the Optimal Control Theory

1. In 1959 J. Kurzweil [4] orignated theory of differential equations with distributions as coefficients. He proved corresponding existence and uniqueness theorems. The theory of such equations gives a convenient tool in the investigation of certain optimal control problems and is closely related to classical boundary problems and problem of periodic solutions. In the present paper we are concerned with a simple case of the second order equation.

2. Let R be the real line. By C we denote the space of all continuous functions defined on R. A sequence $\{x_n\} \subset C$ will be called convergent to x, if it is uniformly convergent to x in every compact $K \subset R$.

By M we denote the space of all distributions on R of order zero (measures in R). A sequence $\{p_n\} \subset M$ will be called convergent to p if $\langle p_n, \varphi\rangle \to \langle p, \varphi\rangle$ for every $\varphi \in M$ with a compact support.

For $p \in M$ and an interval $\Delta \subset R$ let $\|p\|_\Delta$ denote the total variation over Δ of the measure corresponding to p. If p is a locally Lebesgue integrable function, we have

$$\|p\|_\Delta = \int_\Delta |p(t)|\, dt .$$

Consider a second order differential equation

$$x'' + px + q = 0 , \tag{1}$$

where the coefficients $p, q \in M$ and the unknown function $x \in C$. It is evident that every solution $x \in C$ of Eq. (1) possesses the derivative x' which is the function with locally bounded variation. Therefore the initial data for Eq. (1) may be given by conditions

$$x(a) = r_0 \qquad x'(a-) = r_1 , \tag{2}$$

where $x'(a-)$ denotes the left-hand limit of x at the point a.

Denote by P and Q the left-hand continuous primitives of the distributions p and q satisfying conditions $P(a) = Q(a) = 0$. The initial value problem (1), (2) is equivalent to the problem of finding a solution of the integral equation

$$x(t) = r_0 + r_1(t-a) - \int_a^t (t-s)x(s)\,dP(s) - \int_a^t Q(s)\,ds\,. \tag{3}$$

The existence of a unique solution of (1), (2) or (3) may be proved by the classical method of successive approximations (see [1], sec. 11.2). This solution is a continuous function of the initial data (r_0, r_1) and the coefficients p, q.

We have namely the following

Theorem 1. *Let the sequences* $\{r_{0n}\}, \{r_{1n}\} \subset R$ *and* $\{p_n\}, \{q_n\} \subset M$ *be convergent respectively to* r_0, r_1, p, q. *Suppose that there exists* $\varepsilon > 0$ *such that*

$$(a-\varepsilon, a) \cap \operatorname{supp} p = (a-\varepsilon, a) \cap \operatorname{supp} q = \varnothing\,. \tag{4}$$

Then the solutions of problem

$$x'' + p_n x + q_n = 0$$

$$x(a) = r_{0n}\,, \qquad x'(a-) = r_{1n}$$

form a sequence $\{x_n\} \subset C$ *which is convergent to a solution of problem* (1) *and* (2).

Proof. Denote respectively by P, Q, P_n, Q_n the left-hand continuous primitives of the distributions p, q, p_n, q_n satisfying conditions $P(a) = Q(a) = P_n(a) = Q_n(a) = 0$. Then the functions x_n satisfy the integral equation

$$x_n(t) = r_{0n} + r_{1n}(t-a) - \int_a^t (t-s)x_n(s)\,dP_n(s) - \int_a^t Q_n(s)\,ds\,. \tag{5}$$

Setting $u_n = x_n - x$ and substracting (3), from (5) we get

$$u_n(t) = \varepsilon_n(t) - \int_a^t (t-s)\,u_n(s)\,dP_n(s) \tag{6}$$

where

$$\varepsilon_n(t) = r_{0n} - r_0 + (t-a)(r_{1n} - r_1) - \int_a^t \big(Q_n(s) - Q(s)\big)\,ds + $$
$$+ \int_a^t (t-s)x(s)\,d\big(P(s) - P_n(s)\big)\,.$$

By (6) we have obviously $P_n(t) = P(t)$ and $Q_n(t) = Q(t)$ for $t \in (a-\varepsilon, a]$. This implies, by $p_n \to p$, $q_n \to q$, that $P_n(t) \to P(t)$, $Q_n(t) \to Q(t)$ almost everywhere and that the functions P_n and Q_n are locally equibounded. Since x is an absolutely continuous function, $\varepsilon_n(t) \to 0$ almost uniformly.

Applying Gronwall lemma (see [1], p. 455) to (6) we obtain

$$|u_n(t)| \leqslant |\varepsilon_n(t)| \exp\{|t-a|\, \|p\|_{[a,t]}\}\,.$$

On other hand, under our assumptions, the functions $\|p\|_{[a,t]}$ are locally equibounded. Thus, from above inequality it easily follows that $u_n(t) \to 0$ almost uniformly and also $x_n(t) \to x(t)$ almost uniformly which completes the proof.