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On Some Equations in Partially Ordered Spaces

In the present paper we consider some method of successive approximations in partially ordered spaces. By a partially ordered space we shall mean an abstract space with a relation "$\leqslant$" satisfying the axioms introduced in [1].

1. Let Z be a topological, partially ordered space, such that for each non-empty subset P, upperly bounded in Z, there exists in Z $\sup P$ (cf. [1], [2], [3]). The convergence (the limit) of an arbitrary sequence $\{x_n\}$, $x_n \in Z$, and the continuity of mappings, we shall mean as the convergence (the limit) and continuity in the topology given in Z. This topology is such that if $u_n \in Z$, $u_n \leqslant u_{n+1}$, $u_n \leqslant u$ $(n = 1, 2, \ldots)$, then $\sup\limits_n \{u_n\} = \lim\limits_{n\to\infty} u_n$. In the product $Z \times Z$ we introduce the partial ordering in such a way that $(u, v) \leqslant (w, z)$ if $u \leqslant w$ and $v \leqslant z$; the topology in $Z \times Z$ is introduced in the ordinary way as the topology defined by the use of the topology in Z.

Let $A = A(u, v)$ be a mapping of $Z \times Z$ into Z, such that

(1.1) if $(u_n, v_n) \leqslant (u_{n+1}, v_{n+1})$ and $(u_n, v_n) \to (u, v)$ as $n \to \infty$, then $A(u_n, v_n) \to \to A(u, v)$ as $n \to \infty$,

(1.2) if $v \leqslant z$, then $A(u, v) \leqslant A(u, z)$ for each $u \in Z$,

(1.3) there exists $B = B(u)$ $(B\colon Z \to Z)$, such that

(a) $A(u, v) \leqslant B(u)$ for $u, v \in Z$,

(b) if $u_n \leqslant u_{n+1}$ and $u_{n+1} \leqslant B(u_{n+1})$ or $u_{n+1} \leqslant B(u_n)$ for $n = 1, 2, \ldots$, then there exists $u \in Z$, such that $u_n \leqslant u$.

(1.4) for each $v \in Z$, there exists $u \in Z$, $u \geqslant v$, such that $u = A(u, v)$.

2. Theorem 1. *If the space Z and the mapping A satisfy the conditions mentioned in* § 1, *then there exists a solution u of the equation*

$$u = A(u, u)\,. \tag{2.1}$$

This solution is given as the limit of a sequence of successive approximations.

Proof. Let u_0 be an arbitrary element of Z. In view of (1.4), there exists $u_1 \in Z$, $u_1 \geqslant u_0$, such that

$$u_1 = A(u_1, u_0)\,. \tag{2.2}$$

From (1.2) it follows that

(2.3) $$u_1 \leqslant A(u_1, u_1) .$$

From the assumptions concerning Z in A (in particular from (1.3)), it follows that there exists the maximal solution u_2 of the equation

(2.4) $$u = A(u_1, u)$$

satisfying $u_2 \geqslant u_1$ (cf. [2], [3]).

Hence we have

(2.5) $$u_2 = A(u_1, u_2) .$$

For u_2 there exists u_3, $u_3 \geqslant u_2$, such that

(2.6) $$u_3 = A(u_3, u_2) .$$

In that way we construct a sequence $\{u_n\}$, such that

(2.7) $$u_{2n+1} = A(u_{2n+1}, u_{2n}) \quad (n = 0, 1, 2, \ldots)$$

(2.8) $$u_{2n} = A(u_{2n-1}, u_{2n}) \quad (n = 1, 2, \ldots) .$$

Obviously, the sequence $\{u_n\}$ is nondecreasing, and moreover it is bounded, because:

$$u_{2n+1} \leqslant B(u_{2n+1}) , \; u_{2n} \leqslant B(u_{2n-1}) \quad (n = 1, 2, \ldots) .$$

Hence $\{u_n\}$ is convergent to some $u \in Z$. Of course $\lim u_n = \lim u_{2n} = \lim u_{2n+1} = u$. By (1.1) we can write:

$$u = \lim u_{2n+1} = \lim A(u_{2n+1}, u_{2n}) = A(\lim u_{2n+1}, \lim u_{2n}) = A(u, u) ,$$

$$u = \lim u_{2n} = \lim A(u_{2n-1}, u_{2n}) = A(\lim u_{2n-1}, \lim u_{2n}) = A(u, u) .$$

Hence u is a solution of (2.1).

3. Remark 1. It is easy to observe, that in the preceding section, we proved that each sequence $\{u_n\}$ which has been constructed by the above procedure introduced in § 2, with an arbitrary initial u_0, is convergent to a solution of (2.1). Notice, that for u_0 we can choose u_1 (for u_2 we can choose u_3, etc.) in a way, which (generally) is not uniquely determined; if we have u_{2n}, then we can choose u_{2n+1} in an arbitrary way, from the set of all u satisfying the condition $u = A(u, u_{2n})$ (cf. (1.4)).

Remark 2. If $B = B(u)$ is a decreasing function, then each nondecreasing sequence satisfying the condition (1.3)–(b) is bounded.

4. Let us consider an example from the theory of functional equations. In this section the relation "$<$" in the set of functions defined in the interval $[a, b]$ or in $[a, b] \times (-\infty, \infty)$ is understood in such a way that: $u < v$ if $u(x) \leqslant v(x)$ for all x and there exists y such that $u(y) \neq v(y)$.

Let us consider $F = F(x, u)$, $G = G(x, u)$ defined in $[a, b] \times (-\infty, \infty)$, continuous with respect to u, such that F is strictly decreasing and G is strictly increasing with respect to u. Let us consider moreover a function $f = f(x)$, $f:[a, b] \to [a, b]$. Assume that

(I) $0 \leqslant G(x, u) \leqslant M(x)$, M is a function defined in $[a, b]$,
(II) for each function $u = u(x)$ satisfying

$$0 \leqslant u < M + \max_x F(x, 0),$$

there exists $y = y(x)$, such that

$$y(x) = F\big(x, y\big(f(x)\big)\big) + G\big(x, u(x)\big) \tag{4.1}$$

and

$$u \leqslant y < M + \max_x F(x, 0). \tag{4.2}$$

From Theorem 1, in view of Remark 2, it follows directly the following

Theorem 2. *If the functions F, G and f fulfill the above assumptions, then there exists in $[a, b]$ a solution $y = y(x)$ of the equation*

$$y(x) = F\big(x, y\big(f(x)\big)\big) + G\big(x, y(x)\big). \tag{4.3}$$

REFERENCES

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