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Remarks on Some Functional Equations and Inequalities

The present paper is the third author's note on the extremal solutions of some functional equations and functions fulfilling certain functional inequalities. The previous papers [3] and [4] discussed the functional equations of the form

$$y(x) = F\{x, y[f(x)]\} \tag{0.1}$$

and

$$y(x) = F\{x, y[f_1(x)], \dots, y^n[f_n(x)]\}, \tag{0.2}$$

where $y^n(x)$ denotes the n-th iteration of unknown function $y = y(x)$, and certain inequalities.

In the present paper we shall give some supplementary remarks concerning the equations (0.1) and (0.2) and inequalities

$$v(x) \leqslant F\{x, v[f(x)]\} \tag{0.3}$$

and

$$v(x) \leqslant F\{x, v[f_1(x)], \dots, v^n[f_n(x)]\}. \tag{0.4}$$

We shall use the results of the other authors' papers [1] and [2] and the notations introduced in [3], [4], [5]. In particular, by the maximal (minimal) solution of (0.1) (or (0.2)) in some class K, we shall mean a function $y \in K$, satisfying (0.1) (or (0.2)), such that if $z \in K$ is an arbitrary solution of (0.1) (or (0.2)), then $z \leqslant y$ $(y \leqslant z)$.

1. Let us put $\Psi = \{f\colon f \text{ is a real function defined in } \langle a, b\rangle\}$

$$\Phi = \{f\colon f \in \Psi,\ f(\langle a, b\rangle) \subset \langle a, b\rangle\}.$$

Theorem 1.1. *Let us assume that* $f \in \Phi$ *and* $F = F(x, y)$ *is defined in* $T = \langle a, b\rangle \times \langle c, d\rangle$, F *is increasing with respect to* y, $c \leqslant F(x, y) \leqslant d$ *for each* $(x, y) \in T$.

Then there exist in the class Ψ *the maximal and the minimal solutions of the equation* (0.1).

Theorem 1.2. *If all assumptions of Theorem* 1.1 *are satisfied and* $v = v(x)$ *fulfills the inequality* (0.3), *then*

$$v \leqslant y^*,$$

where $y^* = y^*(x)$ *is the maximal solution of* (0.1).

In order to prove Theorems 1.1 and 1.2 we use Theorems 1 and 2 with Remark 1 from [2] or Theorems A and A' with Remark 4 from [1]. In this case we can use also the theorem of Tarski (Theorem 1 in [6]) which is cited in [2].

2. **Theorem 2.1.** *If all assumptions of Theorem* 1.1 *are satisfied and moreover* F *is continuous in* T *with respect to* (x, y), f *is continuous in* $\langle a, b\rangle$, *then the maximal (minimal) solution of* (0.1) *is upper semi-continuous (lower semi-continuous).*

Proof. We put

$$(2.1)\qquad \begin{aligned} y_0(x) &= d \\ y_{n+1}(x) &= F\{x, y_n[f(x)]\} \qquad n = 0, 1, \ldots \end{aligned}$$

It is easy to see that

1° $\qquad y_n(x)$ is for each n continuous,

2° $\qquad y_{n+1}(x) \leqslant y_n(x)$ for $x \in \langle a, b\rangle$, $n = 0, 1, \ldots$

3° $\qquad c \leqslant y_n(x) \leqslant d$ for $x \in \langle a, b\rangle$, $n = 0, 1, \ldots$

Hence the sequence $\{y_n(x)\}$ is convergent and its limit $y = y(x)$ is an upper semi-continuous function. Going to the limit in the relations (2.1) we have (by the continuity of F)

$$(2.2)\qquad y(x) = F\{x, y[f(x)]\}.$$

On the other hand, for each function $v = v(x)$ satisfying the inequality (0.3), it can be shown by the induction that

$$v(x) \leqslant y_n(x) \text{ for } x \in \langle a, b\rangle \text{ and } n = 0, 1, \ldots$$

Hence $v(x) \leqslant y(x) = \lim y_n(x)$. Thus $y = y(x)$ is the maximal solution of (0.1) in the class Ψ.

In order to prove that the minimal solution of (0.1) is a lower semi-continuous function, we consider the following sequence

$$\begin{aligned} w_0(x) &= c \\ w_{n+1}(x) &= F\{x, w_n[f(x)]\} \qquad n = 0, 1, \ldots \end{aligned}$$

which is increasing and convergent to the minimal solution of (0.1).

Corollary 2.1. *If the assumptions of Theorem* 2.1 *are satisfied and if the solution of* (0.1) *is unique, then it is continuous.*

3. One can prove, considering the sequence of the type (2.1) and using the method given in the section 2, that the following theorem, being a generalization of Theorem 2 of [3], holds:

Theorem 3.1. *If all assumptions of Theorem* 1 *of the paper* [3] *are satisfied and a function* v *fulfills* (0.3), *then*

$$v \leqslant y^*,$$

where $y^* = y^*(x)$ *is the maximal solution of* (0.1) *in the subclass of* Ψ *which contains all functions belonging to* Ψ *and fulfilling the Lipschitz condition with the constant* L *introduced in* [3].

In other words, the maximal solution of (0.1) in Ψ is under the assumptions of Theorem 1 from [3], the Lipschitz function. The similar theorem concerning the minimal solution holds too. Remark 3.1. In [3] was introduced the following condition: $|F| \leqslant C$; here we assume that $c \leqslant F \leqslant d$, but it is easy to see, that the difference between these two conditions is essential.

4. **Theorem 4.1.** *Let us assume the following conditions* (*cf. Theorems* 1 *and* 3 *in* [4]):

1. $F = F(x, y_1, \ldots, y_n)$ *is defined in the set* $S = \langle a, b\rangle \times \langle c, d\rangle^n$ *where* $\langle c, d\rangle \subset \langle a, b\rangle$,
2. $F(S) \subset \langle c, d\rangle$,
3. F *is increasing with respect to each variable,*
4. $f_i \in \Phi$ $(i = 1, \ldots, n)$,
5. *if* $x \leqslant u$, *then* $f_i(x) \leqslant f_i(u)$ *for* $i = 1, \ldots, n$,
6. $|F(x, y_1, \ldots, y_n) - F(\hat{x}, \hat{y}_1, \ldots, \hat{y}_n)| \leqslant M|x - \hat{x}| + \sum_{i=1}^{n} M_i |y_i - \hat{y}_i|$,
7. $|f_i(x) - f_i(\hat{x})| \leqslant N_i |x - \hat{x}|$, $(i = 1, 2, \ldots, n)$,

where the constants M_j *and* N_i $(j = 0, 1, \ldots, n,\ i = 1, \ldots, n)$ *are such that the set* $\Lambda = \left(\lambda : \lambda > 0,\ M_0 + \sum_{i=1}^{n} M_i N_i \lambda^i \leqslant \lambda\right)$ *is non-empty.*

Then the maximal solution y_λ *of* (0.2) *in the class* L_λ *of all functions belonging to* Φ, *increasing and satisfying the condition of Lipschitz with the constant* $\lambda \in \Lambda$, *is such that for each* $v \in \Phi$ *fulfilling* (0.4), *the inequality*

$$v(x') \leqslant y_\lambda(x), \quad x \in \langle a, b\rangle \tag{4.1}$$

holds.

Proof. We put

$$y_0(x) = d \tag{4.2}$$

$$y_{k+1}(x) = F\{x, y_k[f_1(x)], \ldots, y_k^n[f_n(x)]\} \qquad k = 0, 1, \ldots$$

It is easy to see that for $k = 0, 1, \ldots$ and $x \in \langle a, b\rangle$ we have

$$1^\circ \quad y_k \in L_\lambda, \quad c \leqslant y_k(x) \leqslant d$$

$$2^\circ \quad x \leqslant \hat{x} => y_k(x) \leqslant y_k(\hat{x})$$

$$3^\circ \quad y_{k+1}(x) \leqslant y_k(x).$$

Hence the sequence $\{y_k(x)\}$ is uniformly convergent to a function $y_\lambda = y_\lambda(x)$ which fulfills the equation (0.2). It is easy to see that $y_\lambda \in L_\lambda$. On the other hand, for each $v \in \Phi$ satisfying (0.4) we have $v \leqslant y_k$ $(k = 0, 1, \ldots)$. It holds in particular for $v \in L_\lambda$. Hence y_λ is the maximal solution of (0.2) in the class L_λ and the inequality (4.1) holds for each $v \in \Phi$ satisfying (0.4). In other words, if the assumptions of Theorem 4.1 are satisfied, then the maximal solution of (0.2) in Φ exists and it belongs to the class L_λ.

A similar theorem concerning the minimal solution of (0.2) holds too.

Corollary 4.1. *Because* $\lambda \leqslant \mu \Rightarrow L_\lambda \subset L_\mu$ *and in consequence* $y_\lambda \leqslant y_\mu$ *and on the other hand in virtue of the above Theorem* 4.1 *we have* $y_\mu \leqslant y_\lambda$ *for each* $\lambda \in \Lambda$, *then if* $\min \Lambda > 0$, *we can put in the above* $\lambda = \min \Lambda$.

Remark 4.1. Theorem 4.1 is a generalization of Theorem 8 in [4]. In the paper [4] we have proved Theorem 4 concerning the problem of the existence of the maximal solution of (0.2) in the class W of all increasing, convex and bounded functions defined in $\langle a, b\rangle$. The following theorem is an answer to the question of the existence of the minimal solution of (0.2) in W.

Theorem 4.2. *Let us assume the conditions* 1–7 *of Theorem* 4.1 *and the following conditions* (cf. *the assumptions* 9 *and* 10 *of Theorem* 4 *in* [4]):

8. $F\left(\tfrac{1}{2}(x+\hat{x}), \tfrac{1}{2}(y_1+\hat{y}_1), \ldots, \tfrac{1}{2}(y_n+\hat{y}_n)\right) \leqslant \tfrac{1}{2}\left(F(x, y_1, \ldots, y_n)+F(\hat{x}, \hat{y}_1, \ldots, \hat{y}_n)\right),$

9. $f_i\left(\dfrac{x+\hat{x}}{2}\right) \leqslant \dfrac{1}{2}\left(f_i(x)+f_i(\hat{x})\right) \quad (i = 1, \ldots, n).$

Then the equation (0.2) *has the minimal solution in the class* Φ *and this solution belongs to the class* $W^* = W \cap L_\lambda$.

Proof. We put:

$$w_0(x) = c \tag{4.3}$$

$$w_{k+1}(x) = F\{x, w_k[f_1(x)], \ldots, w_k^n[f_n(x)]\}, \qquad k = 0, 1, \ldots$$

The sequence (4.3) is increasing, all functions w_k belong to W^*, $w_k \leqslant d$ for each k. Hence $\{w_k(x)\}$ is convergent to a function $w \in W^*$, which evidently fulfills (0.2). On the other hand, for each $v \in \Phi$, such that

$$v(x) \geqslant F\{x, v[f_1(x)], \ldots, v^n[f_n(x)]\} \tag{4.4}$$

it can be easily proved by the induction with respect to k, that

$$v(x) \geqslant w_k(x) \qquad k = 0, 1, \ldots \tag{4.5}$$

Hence

$$v(x) \geqslant w(x) \quad \text{for} \quad x \in \langle a, b\rangle \tag{4.6}$$

what means that w is the minimal solution of (0.2) in the class Φ because (4.6) holds in particular for each solution of (0.2).

Remark 4.2. We can notice that in this case, the similar conclusion, as in the case discussed in Remark 3.1, holds too.

5. **Theorem** 5.1. *If* $|F(x, y)-F(x, \hat{y})| \leqslant K|y-\hat{y}|$, $K<1$, *then there exists in* Ψ *at most one solution of* (0.1).

Proof. Let $y_i = y_i(x)$ $(i=1,2)$ be two solutions of (0.1). Hence:

$$\begin{aligned} |y_1(x)-y_2(x)| &\leqslant \Big|F\big(x, y_1(f(x))\big)-F\big(x, y_2(f(x))\big)\Big| \leqslant \\ &\leqslant K\cdot\Big|y_1(f(x))-y_2(f(x))\Big| \leqslant \\ &\leqslant K\cdot \sup_{\langle a,b\rangle} |y_1(x)-y_2(x)| \end{aligned}$$

and then $s = \sup\limits_{\langle a,b\rangle} |y_1(x)-y_2(x)|$ fulfills the inequality:

$$s < K\cdot s$$

which implies, in virtue of the assumption $K<1$, that $s=0$.

Corollary 5.1. *If the constant* N *introduced in the assumptions of Theorem* 1 *of* [3] *is* $\geqslant 1$ *and all assumptions of this theorem are satisfied, then there exists exactly one solution of* (0.1).

Theorem 5.2. *If* $|F(x; y_1, \ldots, y_n)-F(x, \hat{y}_1, \ldots, \hat{y}_n)| \leqslant \sum\limits_{i=1}^{n} M_i|y_i-\hat{y}_i|$ *and* $\sum\limits_{i=1}^{n} M_i < 1$, *then* (0.2) *has at most one solution in* Φ.

The proof of the present theorem is similar to the proof of Theorem 5.1.

Corollary 5.2. *If the constants* N_i *introduced in the assumptions of Theorem* 4.1 *are such that* $N_i = N \geqslant 1$ $(i = 1, \ldots, n)$ *and all assumptions of Theorem* 4.1 *are satisfied, then there exists in* Φ *exactly one solution of* (0.2).

As a generalization of Theorem 5.1 one can prove the following:

Theorem 5.3. *Let us assume that the functions* F *and* f *fulfill the assumptions of Theorem* 1.1 (*or respectively*: Theorem 1 of [3], or Theorem 1 (for $n=1$) *of* [4], *or Theorem* 4 (*for* $n=1$) *of* [4]), $F(x, y) \geqslant 0$ *for* $y \geqslant 0$ *and the maximal solution of* (0.1) *in the class* Φ (*or respectively*: *in the class of Lipschitz functions, or in the class of increasing functions, or in the class of increasing and convex functions*) *is identically equal to zero. If* $|G(x, y)-G(x, \hat{y})| \leqslant F(x, |y-\hat{y}|)$, *then there exists at most one solution* $y = y(x)$ *of the equation*

$$y(x) = G\big(x, y\,(f(x))\big)\,. \tag{5.1}$$

Proof. Let y and $\hat{y}$ be two solution of (5.1). Hence for $v = |y-\hat{y}|$ we have the inequality:

$$v(x) \leqslant F\big(x, v(f(x))\big)$$

which in virtue of the previous Theorems 1.2, 2.1 and Theorems 7 and 8 from [4], finishes the proof of Theorem 5.3.

Remark 5.1. An example of the function F satisfying the assumptions of Theorem 5.3 is the following one: $F(x, y) = K\cdot y$, where $K<1$ (see Theorem 5.1).