ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
Nr CLXVII PRACE MATEMATYCZNE, ZESZYT 12 1968

Andrzej Pelczar

On Some Iterative-Differential Equations I

In the present paper we discuss the problem of the existence and uniqueness of solutions in the interval $\langle 0, a\rangle$ of the equation

$$\frac{dy}{dx} = f\big(x, y(x), y\big(y(x)\big)\big) \tag{1}$$

with the initial condition

$$y(0) = c \tag{2}$$

and the question of the convergence of some sequences of successive approximations.

By a solution of the problem (1)–(2) we mean a function $y \in C^1(\langle 0, a\rangle)$ satisfying in $\langle 0, a\rangle$ (1) and (2). Hence, if the function f is continuous, then the problem (1)–(2) is equivalent to the problem of the existence of continuous solution of the integral equation:

$$y(x) = c + \int_0^x f\big(t, y(t), y\big(y(t)\big)\big)\,dt\,. \tag{3}$$

1. Let $f = f(x, y, u)$ be defined and continuous in $\langle 0, a\rangle \times (-\infty, \infty) \times \times (-\infty, \infty)$ and let

$$|f(x, y, u) - f(x, \hat{y}, \hat{u})| \leqslant M\cdot|y - \hat{y}| + N\cdot|u - \hat{u}| \tag{4}$$

$$|f(x, y, u)| \leqslant K \tag{5}$$

for $(x, y, u), (x, \hat{y}, \hat{u}) \in \langle 0, a\rangle \times (-\infty, \infty) \times (-\infty, \infty)$.

Theorem 1. *If $N\cdot a < e^{-(M+NK)Ka}$, and f satisfies the above conditions, then there exists at most one solution of the problem* (1)–(2).

Proof. Let $z = z(x)$ and $w = w(x)$ be two solutions of the problem (1)–(2). Hence we have:

$$|z(x)-w(x)| \leqslant \int_0^x \Big(M|z(t)-w(t)|+N\big|z\big(z(t)\big)-w\big(w(t)\big)\big|\Big)\,dt \leqslant$$

$$\leqslant M\int_0^x |z(t)-w(t)|\,dt+N\int_0^x \big|z\big(z(t)\big)-z\big(w(t)\big)\big|\,dt+$$

$$+N\int_0^x \big|z\big(w(t)\big)-w\big(w(t)\big)\big|\,dt \leqslant$$

$$\leqslant (M+NK)\int_0^x |z(t)-w(t)|\,dt+N\int_0^x \big|z\big(w(t)\big)-w\big(w(t)\big)\big|\,dt\,.$$

If we put $u(x) = |z(x)-w(x)|$ and $L = M+NK$, $\hat{K} = 2K$, then

$$u(x) \leqslant L\int_0^x u(t)\,dt+N\int_0^x u\big(w(t)\big)\,dt \tag{6}$$

and in consequence

$$u(x) \leqslant N\hat{K}a^2+L\int_0^x u(t)\,dt\,. \tag{7}$$

From (7) and well known theorems concerning the differential (integral) inequalities, it follows that

$$u(x) \leqslant N\hat{K}a^2 e^{Lx}\,. \tag{8}$$

From (6) and (8) it follows that

$$u(x) \leqslant L\int_0^x u(t)\,dt+N\int_0^x N\hat{K}a^2 e^{Lw(t)}\,dt \leqslant L\int_0^x u(t)\,dt+N^2\hat{K}a^3 e^{LKa},$$

and in the consequence $u(x) \leqslant N^2\hat{K}a^3 e^{LKa} e^{Lx}$. It is easy to show by the induction with respect to n, that

$$u(x) \leqslant \hat{K}ae^{La}N^n a^n e^{(n-1)LKa} \leqslant \hat{K}ae^{L(a-K)}N^n a^n e^{nLKa} \tag{9}$$

for $n = 2, 2, \ldots$. Since, from the assumptions it follows that $Na < e^{-LKa}$, then $Na\cdot e^{LKa} < 1$ and $u(x)$ is upperly bounded by the sequence which tends to zero as $n \to \infty$. Since $u(x) \geqslant 0$, it must be equal to zero, what ends the proof of Theorem 1.

2. Let us suppose, that

$$c+Ka \leqslant a\,, \qquad c \geqslant Ka \tag{10}$$

and let us consider the following sequences:

$$y_{n+1}(x) = c+\int_0^x f\Big(t,\, y_n(t),\, y_n\big(y_n(t)\big)\Big)\,dt \tag{11}$$

$$w_{n+1}(x) = c+\int_0^x f\Big(t,\, w_n(t),\, w_n\big(w_{n+1}(t)\big)\Big)\,dt \tag{12}$$

$$z_{n+1}(x) = c+\int_0^x f\Big(t,\, z_{n+1}(t),\, z_n\big(z_{n+1}(t)\big)\Big)\,dt \tag{13}$$

$(n=0,1,2,\ldots)$, where $y_0(x)$, $w_0(x)$, $z_0(x)$ are some (arbitrary) fixed functions of class C^1, mapping $\langle 0,a\rangle$ into $\langle 0,a\rangle$, such that $|y_0'(x)|, |w_0'(x)|, |z_0'(x)| \leqslant K$. If w_n (resp. z_n) is given, then w_{n+1} (resp. z_{n+1}) is the solution of (12) (resp. (13)). From the well known theorems concerning ordinary differential equations, it follows that under our assumptions, such solution of the equation (12) (resp. (13)) with respect to unknown function w_{n+1} (resp. z_{n+1}) exists and is unique. From the condition (10) it follows that we can make the operation of the iteration; hence the definitions of the sequences (11)–(13) are correct. It is easy to see that $0 \leqslant y_n(x),\ w_n(x),\ z_n(x) \leqslant Ka+c$ and $|y_n(x)-y_n(\hat{x})|$, $|w_n(x)-w_n(\hat{x})|$, $|z_n(x)-z_n(\hat{x})| \leqslant K\cdot|x-\hat{x}|$, for $n=0,1,2,\ldots$ and $x,\hat{x}\in\langle 0,a\rangle$. Hence there exist $\{y_{a_n}\}$, $\{w_{a_n}\}$, $\{z_{a_n}\}$ uniformly convergent in the interval $\langle 0,a\rangle$.

3. **Theorem 2.** *Let the assumptions of Theorem* 1 *and the condition* (10) *hold. If* $La+Na<1$, *then the sequences* (11)–(13) *converges uniformly to the* (*unique*) *solution* $y=y(x)$ *of the problem* (1)–(2).

Proof. We put

$$Y_n = \max_{\langle 0,a\rangle} |y_n(x)-y_{n-1}(x)|$$

$$W_n = \max_{\langle 0,a\rangle} |w_n(x)-w_{n-1}(x)|$$

$$Z_n = \max_{\langle 0,a\rangle} |z_n(x)-z_{n-1}(x)| \qquad n=1,2,\ldots$$

It is easy to see that

$$Y_n \leqslant (L+N)a\cdot Y_{n-1} \leqslant \big((L+N)a\big)^{n-1}\cdot Y_1$$

$$W_n \leqslant \frac{(M+N)a}{1-NKa}\cdot W_{n-1} \leqslant \left[\frac{(M+N)a}{1-NKa}\right]^{n-1}\cdot W_1$$

$$Z_n \leqslant \frac{Na}{1-La}\cdot Z_{n-1} \leqslant \left[\frac{Na}{1-La}\right]^{n-1}\cdot Z_1 .$$

From the inequality $La+Na<1$ it follows that $(L+N)a<1$, $NKa<1$ and $(M+N)a<1-NKa$ and moreover $La<1$ and $Na<1-La$. Hence the sequences $\{Y_n\}$, $\{W_n\}$, $\{Z_n\}$ and so $\{|y_n(x)-y_{n-1}(x)|\}$, $\{|w_n(x)-w_{n-1}(x)|\}$, $\{|z_n(x)-z_{n-1}(x)|\}$ tends uniformly to zero as $n\to\infty$. In consequence if a subsequence $\{y_{a_n}(x)\}$ (resp. $\{w_{a_n}(x)\}$ or $\{z_{a_n}(x)\}$) is uniformly convergent to $\hat{y}(x)$ (resp. $\hat{w}(x)$ or $\hat{z}(x)$), then the subsequence $\{y_{a_n-1}(x)\}$ $\big(\{w_{a_n-1}(x)\}$, $\{z_{a_n-1}(x)\}\big)$ is uniformly convergent to the same limit. Let $\{y_n^*(x)\}$, $\{w_n^*(x)\}$, $\{z_n^*(x)\}$ be arbitrary, uniformly convergent subsequences of the sequences — respectively — $\{y_n(x)\}$, $\{w_n(x)\}$ and $\{z_n(x)\}$. Passing to the limits in (11), (12) and (13) (for the sequences with the asterisks) we obtain as the conclusion, that $\lim y_n^*(x)$, $\lim w_n^*(x)$ and $\lim z_n^*(x)$ are solutions of (3), which is equivalent to the problem (1)–(2). Since the solution of (3) is unique, then $\lim y_n^*(x) = \lim w_n^*(x) = \lim z_n^*(x)$ and moreover each subsequence of $\{y_n\}$ $\big(\{w_n(x)\}$, $\{z_n(x)\}\big)$ uniformly convergent, must be convergent to this solution. Hence the whole sequences (11), (12) and (13) are uniformly convergent to the unique solution of the equation (3).

4. Let us suppose the assumptions of Theorem 2 and moreover, suppose that $y_0 = w_0 = z_0$.

We put

$$P_n = \max_{\langle 0,a\rangle} |y_n(x) - y(x)| \tag{14}$$

$$Q_n = \max_{\langle 0,a\rangle} |w_n(x) - y(x)| \tag{15}$$

$$R_n = \max_{\langle 0,a\rangle} |z_n(x) - y(x)| \tag{16}$$

where $y = y(x)$ is the unique solution of (3).

Of course $P_0 = Q_0 = R_0 = D$. It is easy to see that

$$P_n \leqslant \big(a(M+N+NK)\big)^n D \overset{\text{df}}{=} P_n^* \tag{17}$$

$$Q_n \leqslant \left[\frac{a(M+N)}{1-aNK}\right]^n \cdot D \overset{\text{df}}{=} Q_n^* \tag{18}$$

$$R_n \leqslant \left[\frac{aN}{1-a(M+NK)}\right]^n \cdot D \overset{\text{df}}{=} R_n^* . \tag{19}$$

From the above assumptions it follows that

$$M+N+NK > \frac{M+N}{1-aNK} > \frac{N}{1-a(M+NK)}$$

Hence

$$P_n^* \geqslant Q_n^* \geqslant R_n^* . \tag{20}$$

Of course this relation give no information concerning the relations between P_n, Q_n and R_n. This relation (20) gives however an information on the type of convergence to zero of the sequences with asterisks, being some estimates of the sequences of the maximal differences between y_n, w_n and dz_n and the solution y. These estimates are dependent on the constants M, N, K and a only, and in the consequence, they are some estimates for the class of functions defined by these constants.

5. Let us notice, that the above reasoning concerning the equation (1) with the initial condition (2) in the interval $\langle 0, a\rangle$, one can make, without any essential changes, with respect to (1) and the initial condition $y(a_1) = c$ in an arbitrary interval $\langle a_1, a_2\rangle$.

The author would like to express his thanks to A. Lasota for his valuable remarks.