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A Note on the Green's Function

Let D be a bounded domain in Euclidean m-space R^m; let the partial differential operator L defined by

$$Lu = \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)\,u$$

be elliptic there. Denote by $\tilde{D}$ the set $\tilde{D} = (\bar{D} \times \bar{D}) \setminus (\partial D \times \partial D)$.

A function $G = G(x, y)$ will be called a Green function for the equation $Lu = 0$ with respect to the domain D if

(i) $G(x, y)$ is continuous on $\tilde{D}$, for $x \neq y$,
(ii) $G(x, y) = 0$ for $x \in \partial D$, $y \in D$;
(iii) the integral

$$I_G(x) = \int_D G(x, y) f(y)\, dy$$

is of class C^2 on D continuous on $\bar{D}$ and satisfies the identity

$$L[I_G(x)] = -f(x)$$

for all $f(x)$ in $C^1(D)$.

In the sequel we shall use

$$A(x, y) = \frac{\sum_{i=1}^{m} (y_i - x_i)^2 \left[\sum_{i=1}^{m} a_{ii}(x) - \sum_{i,j=1}^{m} \frac{\partial a_{ij}}{\partial x_j} \cdot (y_i - x_i) + \frac{c(x) \sum_{i=1}^{m} (y_i - x_i)^2}{-m+3+a} \right]}{\sum_{i,j=1}^{m} a_{ij}(x)(y_i - x_i)(y_j - x_j)}, \quad 0<a<1.$$

In the special case of Laplace's equation $A(x, y) = m$.

Theorem. *Let us assume that the coefficients $a^{ij}(x)$ of L are twice continuously differentiable on D, $c(x)$ is continuous function obeying $c(x) \leqslant 0$ on D and $m \geqslant 4$. Suppose that for any fixed point y_0 of D there exist a sphere $K(y_0, \varrho) \Subset D$ and a constant $\varepsilon (0 < \varepsilon < 1)$ such that*

$$\mathcal{A}(x, y) > m-2+\varepsilon$$

for x and y in $K(y_0, \varrho)$. Then there exist two constants $M > 0$ and a $(0 < a < 1)$ such that

$$G(x, y) \leqslant U(x, y)+Mr^{-m+3+a}$$

for x and y in $K(y_0, \bar{\varrho})$, where

$$U(x, y) = 2^{\frac{m-2}{2}} \frac{1}{(m-2)\omega_m \sqrt[4]{A(x)}\sqrt[4]{A(y)}} \times \\ \times \left\{ \sum_{i,j=1}^{m} [A_{ij}(x)+A_{ij}(y)](y_i-x_i)(y_j-x_j) \right\}^{\frac{-m+2}{2}},$$

$r^2 = \sum_{i=1}^{m} (y_i-x_i)^2$, *$[A_{ij}(x)]$ is the inverse matrix to $[a_{ij}(x)]$, ω_m is the area of the unit sphere in R^m, $A(x)$ is the determinant of the matrix $[A_{ij}(x)]$, and $0 < \bar{\varrho} < \varrho$.*

Proof. Observe that under our assumptions concerning the coefficients, $U(x, y)$ is the Levi's function (see [1]) and has the property that

$$L_x U(x, y) = 0(r^{-m+2}), \text{ as } r \to 0,$$

where L_x denotes the operator L applied with respect to the variables x. Consider the function

$$\Phi(x, y) = -U(x, y)-Mr^{-m+3+a}+G(x, y).$$

The constants M, a will be defined later. Let $I_\Phi(x)$ denote the integral

$$I_\Phi(x) = \int_D \Phi(x, y)f(y)dy,$$

where $f(y)$ is in class $C^1(D)$. Then the function $I_\Phi(x)$ is in class $C^2(D)$ (for details see [1]). From the Poisson's theorem ([1], formula 13.7), the theorem 12. VIII of [1] and from the definition of Green's function we obtain (cf [2])

$$L[I_\Phi(x)] = \int_D L_x[-U(x, y)-Mr^{-m+3+a}]f(y)dy.$$

On the other hand

$$L_x[-U(x, y)] = 0(r^{-m+2}), \text{ as } r \to 0$$

and

$$L_x(Mr^{-m+3+a}) = M(-m+3+a)r^{-m-1+a} \sum_{i,j=1}^{m} a_{ij}(x)(y_i-x_i)(y_j-x_j) \times \\ \times [(-m+1+a)+\mathcal{A}(x, y)].$$

For instance we can fix $\alpha = 1 - \frac{\varepsilon}{2}$. From

$$L_x\left[Mr^{-m+4-\frac{\varepsilon}{2}}\right] = 0\left(r^{-m+2-\frac{\varepsilon}{2}}\right), \text{ as } r \to 0,$$

there exists a constant $M > 0$ such that

$$L_x\left[-U(x,y) - Mr^{-m+4-\frac{\varepsilon}{2}}\right] \geqslant 0$$

for x and y in $K(y_0, \varrho)$, $x \neq y$. As in the theorem 3 of [2], let $f(y)$ be a non--negative $C^1(D)$ function vanishing outside $K(y_0, \bar{\varrho})$ and positive on its interior. Then we have

$$L\Big[\int_{K(y_0,\bar{\varrho})} \Phi(x,y) f(y)\,dy\Big] \geqslant 0 \tag{1}$$

in $K(y_0, \varrho)$. On the other hand let M be chosen so large that the inequality

$$-U(x,y) - Mr^{-m+3+\alpha} + G(x,y) \leqslant 0$$

holds for $x \in \partial K(y_0, \varrho)$, $y \in K(y_0, \bar{\varrho})$. Hence

$$\int_{K(y_0,\bar{\varrho})} \Phi(x,y) f(y)\,dy \leqslant 0 \tag{2}$$

on $\partial K(y_0, \varrho)$. From (1), (2) and from the Hopf theorem (see [1], theorem 3.II) it follows

$$\int_{K(y_0,\bar{\varrho})} \Phi(x,y) f(y)\,dy \leqslant 0$$

on $K(y_0, \varrho)$. Then $\Phi(x,y) \leqslant 0$ for $x \in K(y_0, \varrho)$, $y \in K(y_0, \bar{\varrho})$. Thus the proof of the theorem is complete.

Remark. We note that the preceding proof remains valid if $m \geqslant 4$. Does there exists a number $M > 0$ such that

$$G(x,y) \leqslant U(x,y) + M$$

for $x \in K(y_0, \varrho)$, $y \in K(y_0, \varrho)$ in the case $m = 3$?

REFERENCES

[1] C. Miranda, *Equazioni alle derivate parziali di tipo ellittico*, Berlin 1955.

[2] B. Szafirski, *On the Green's function for second order elliptic differential equations*, Colloquium Mathematicum 16 (1967), 117—121.