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Algebraic Objects

Introduction. In the present paper we define the notion of an algebraic object as some type of the algebraic operator structure. Then we give a series of definitions bounded with this notion and we prove some propositions and theorems about the algebraic objects. From the algebraic point of view this paper may be regarded as a treatise about such operator structures which have only exterior operations.

The standpoint to definition of algebraic object was the notion of geometric object (cf. [1], [4]), and more exactly the notion of so-called „abstract” geometric object (cf. [4], p. 24). The acting of a transformation group G on the fibre X of this object may be treated as an exterior operation on X with G as the set of operators. In the paper we consider operator structures which are more general than these one which we obtain from geometric objects.

Moreover, algebraic objects may serve as algebraic models of abstract automatic machines (cf. [3]). We shall deal with this problem more exactly in a separate paper.

We take all the fundamental terms, notions and theorems of the theory of algebraic structures used in this paper after Bourbaki [2].

I. The notion of algebraic object

1. Let A be an arbitrary set. We denote by (A, g_i, f_j, F_j) $i = 1, \dots; j = 1, \dots$ the algebraic structure defined on A by interior operations g_i

$$g_i : A^2 \ni (a, b) \rightarrow g_i(a, b) \in A$$

and exterior operations f_j

$$f_j : F_j \times A \ni (a, a) \rightarrow f_j(a, a) \in A$$

with the operator sets F_j . The results of the above operations we shall briefly write multiplicatively: $g(a, b) = ab$ and $f(a, a) = aa$ and we shall call them in both the cases the „products”.

Definition 1. By a *pure operator structure* we shall mean an algebraic structure which have only exterior operations.

Definition 2. A multiplicative system F with one interior operation defined for some ordered pairs $(\alpha, \beta) \in F \times F$ will be called a *semi groupoid* if this operation satisfies the following axioms:

(a) If in the equation

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma$$

on one of its sides or both the products $\beta\gamma$ and $\alpha\beta$ are defined, then both sides of the equation are defined and the equality holds.

(b) To every element a of F there exists exactly one left unit ε_a and exactly one right unit δ_a such that

$$\varepsilon_a a = a \delta_a = a.$$

(c) If the product $\alpha\beta$ is defined then $\delta_\alpha = \varepsilon_\beta$.

Definition 3. A semigroupoid will be called a *groupoid* * if, in addition to axioms (a)-(c), also the following condition holds

(d) To every element a there exists exactly one element a^{-1} (inverse to a) such that

$$aa^{-1} = \varepsilon_a, \quad a^{-1}a = \delta_a.$$

We omit the simple proof of the following proposition.

Proposition 1. If the multiplication $(\alpha, \beta) \rightarrow \alpha\beta$ is defined on the whole Cartesian product $F \times F$ then the semigroupoid is a semigroup with the unit-element and the groupoid is a group.

1. The fundamental notions

Definition 4. A pure operator algebraic structure $\Omega = (X, F)$ ** with the basic set X and the set of operators F will be called a *left algebraic object* over F if the exterior multiplication $(\alpha, x) \rightarrow \alpha x$ defined for some pairs $(\alpha, x) \in F \times X$ satisfies the following axioms:

(A) F has a structure at least of a semigroupoid.

(B) The exterior multiplication is associative in the following sense: if the products αx and $\alpha\beta$ are defined then both sides of the equation

$$\alpha(\beta x) = (\alpha\beta)x$$

are defined and this equality holds.

(C) Any unit is a neutral operator, i.e. if αx is defined then $\varepsilon_\alpha x$ and $\delta_\alpha x$ are defined and the following equalities hold

$$\varepsilon_\alpha x = \delta_\alpha x = x.$$

* There exist also another definitions of groupoid, cf. [6].

** In the following we shall omit the sign of exterior multiplication f in the notation (X, F, f) (as in section 1).

A pure operator structure (X, F) is said to be a *right algebraic object* if it satisfies the axioms (A) and (C) and the following conditional associative law.

(B') If the products βx and βa are defined, then both sides of the equation

$$a(\beta x) = (\beta a)x$$

are defined and this equality holds.

The basic set X will be called a fibre of the object. If the exterior multiplication is defined for all pairs $(a, x) \in F \times X$ then the object will be called *non-singular*, in the opposite case — *singular*. For fixing our attention we shall further consider only the left algebraic objects and they will be termed briefly algebraic objects. All propositions proved for these objects will be valid also, after some formal changes, for right objects.

Proposition 2. *Any algebraic objects over a semigroup with a unit (thus also over a group) is non-singular.*

Proof. Let $a \in F$ and $x \in X$. There exists $\beta \in F$ such that βx is defined*. Since every unit is equal to the unit of semigroup F , thus $\varepsilon_\beta = \varepsilon$ and in virtue of neutrality condition (C) εx is defined and $\varepsilon x = x$. Then, since εx and $a\varepsilon$ are defined, also, in view of (C), $ax = (a\varepsilon)x$ is defined, what completes the proof of proposition.

Definition 5. By a *multiple algebraic object* over F_1, \dots, F_m , $m > 1$, we shall mean an operator structure (X, F_1, \dots, F_m) with the fibre X , with the sets of operators F_1, \dots, F_m and with m exterior operations each of them satisfying the axioms (A), (B), (C) or (A), (B'), (C).

Formally, any multiple algebraic object may be easily represented by a single algebraic object (X, F) **.

Definition 6. Two algebraic objects are said to be *homologous* if they have the same set of operators.

Let be given two homologous algebraic objects

$$(1) \quad \Omega = (X, F)$$

and

$$(2) \quad \theta = (Y, F).$$

The mapping

$$(3) \quad h: X \rightarrow Y$$

is called a *homomorphism* of the object (1) in the object (2) if it is a homomorphism of the algebraic structure (X, F) in the structure (Y, F) . Thus h is a homomorphism, if, whenever product ax is defined, then also product $ah(x)$ is defined and the following equality holds

$$(4) \quad h(ax) = ah(x) \quad ***.$$

* We shall always assume that to every element x there exists at least one operator which acts on x . The elements which do not have this property can be left out from the object.

** It is sufficient to take F as the direct sum (join) of the structures F_j . Then if $a \in F$, there exists exactly one set F_j , such that $a \in F_j$, and product ax has a simple meaning.

*** In general, the sign ax denotes another exterior multiplication as the sign $ah(x)$.

Definition 7. An algebraic object (2) is said to be a *concomitant* of an object (1) if there exists an epimorphism (i.e. homomorphism onto) of (1) onto (2).

Two object will be called *similar* if each of them is a concomitant of the other.

Definition 8. Two isomorphic algebraic objects will be called *equivalent*.

2. The factor-objects

Let R be an equivalence relation defined on the fibre X of the object (1). We say this relation is conformable with the exterior multiplication on X if, whenever products ax and ax' are defined, then the following implication holds

$$xRx' \Rightarrow (ax)R(ax'),$$

where xRx' means that the elements x, x' are congruent according to R .

An equivalence relation which is conformable with the exterior multiplication on X will be called *conformable with the object*.

Definition 9. Let R be such a relation according to the object (1). By the term *factor-object* $\Omega^* = \Omega/R$ we shall mean an algebraic objects which represents a factor-structure of the algebraic structure (X, F) by the relation R .

The fibre of the factor-object Ω^* is the set $X^* = X/R$ of all the equivalence classes $[x]$, where $x \in X$. Object Ω^* has the structure which is induced by the structure of Ω . The induced exterior operation in Ω^* is defined as follows: For arbitrary a and x , whenever product ax is defined, then also product $a[x]$ is defined and by definition

$$(5) \quad a[x] = [ax].$$

The mapping $k: X \ni x \rightarrow [x] \in X^*$ is a homomorphism of object Ω onto factor-object Ω^* . It is called a *canonical (or natural) homomorphism*.

Let θ be a concomitant of Ω and let h denotes an epimorphism Ω onto θ . We can define on the fibre X of Ω the following equivalence relation denoted by (h)

$$(6) \quad x(h)x' \Leftrightarrow h(x) = h(x').$$

This relation is conformable with the object Ω . In fact, if $x(h)x'$ implies $h(x) = h(x')$ and if products ax, ax' are defined, then we have in view of (4)

$$h(ax) = ah(x) = ah(x') = h(ax').$$

Thus $(ax)(h)(ax')$ holds, q.e.d.

Definition 10. We say that a homomorphism (3) satisfies the condition (W) if for every a, x , whenever product $ah(x)$ is defined, then there exists an element x' contained in $h^{-1}(x)$ such that ax' is defined.

Theorem 1. (The fundamental theorem about concomitants). *If an algebraic object (2) is a concomitant of an object (1) and a homomorphism (3) satisfies the condition (W), then the object (2) is equivalent with the factor-object $\Omega/(h)$, where (h) is the relation defined by the formula (6).*

Proof. We shall give an isomorphism $\Omega/(h)$ onto θ . It is easily seen that the mapping g defined as follows

$$(7) \quad g([x]) = h(x)$$

establishes a one-to-one correspondence between the fibre X^* of the object $\Omega/(h)$ and the fibre Y of θ . Moreover, if product $a[x]$ is defined, then, according to the definition (5), it must exist an element $x' \in [x]$ such that ax' is defined and it holds

$$g(a[x]) = g([ax']) = h(ax') = ah(x') = ah(x) = ag([x]).$$

That means that g is an homomorphism. On the other hand, if $ah(x)$ is defined then, in virtue of condition (W), there exists an element x' in the class $[x] = h^{-1}(x)$ such that ax' is defined. Thus we have

$$g^{-1}(ah(x)) = g^{-1}(h(ax')) = [ax'] = a[x] = ag^{-1}(h(x)),$$

which means that g^{-1} is a homomorphism. Thus the mapping g is an isomorphism, q.e.d.

In the case when the object (1) is non-singular, condition (W) is satisfied trivially and the above theorem corresponds immediately to the fundamental theorem about homomorphisms of algebraic structures. In this case the theorem about concomitants takes the following form:

Theorem 1'. *Every concomitant of an non-singular algebraic object is equivalent to one of its factor-objects.*

3. Subobjects

Definition 11. Any substructure of the structure (X, F) will be called a *subobject* of the object (1).

Let U be an arbitrary subset of X and H such one of F . By HU we shall mean either an empty symbol if do not exist elements $a \in H$ and $x \in U$ such that ax is defined, or the set of all defined products ax where $a \in H$, $x \in U$. If U consists of one element x , then instead $H\{x\}$ we shall write Hx .

Definition 12. A subset U of X is said to be *stable* if $FU \subset U$ (according to footnote 3 FU is a non-empty set).

An element $x \in X$ is called an *invariant* if $\{x\}$ is a stable subset of X . An algebraic object is called a scalar if any its element is an invariant.

The scalars as independent algebraic objects are of course not interesting, but they are of great interest as concomitants of other objects — s.c. *scalar concomitants*. The classification of these ones is one of the most important problems in the theory of geometric objects (cf. [7]).

It is easily seen that the restriction of exterior operation to any stable subset induces in this set a structure of algebraic object with the same set of operators. This new algebraic object is thus a subobject of the given object. Conversely, the fibre of any subobject is a stable subset. Thus we can formulate the following:

Proposition 3. *A subset $U \subset X$ is the fibre of a subobject of the object (1) if and only if U is a stable subset of X .*

Definition 13. An algebraic object will be called *reducible* if it has at least one proper subobject, and *irreducible* in the opposite case.

Definition 14. The set FU will be called a *trace* of the subset $U \subset X$ and denoted by S_U . The trace of an one-element subset $\{x\}$ we denote by S_x .

Lemma 1. *If $x \in S_x$ then $S_x \subset S_x$.*

In fact, in view of associativity condition, if ax is defined then $F(ax) = (Fa)x \subset Fx$. Thus if $x' \in S_x$ then there exists an operator a such that $x' = ax$. From this $S_x = Fx' = F(ax) \subset Fx = S_x$, q.e.d.

It follows from the above lemma that the trace of any element is a stable subset. Since the union of an arbitrary family of stable subsets is again a stable subset, thus we can formulate the following proposition.

Proposition 4. *The trace of any subset U of the fibre X is a stable subset of X and it defines a subobject of the object (X, F) . This is the smallest subobject containing U . It will be called a subobject generated by U .*

The fact that the object generated by a subset U contains U follows from the neutrality condition (C). Namely, for any $x \in U$ there exists $a \in F$ such that ax is defined, hence product $\varepsilon_a x$ is defined and $\varepsilon_a x = x$, thus $x \in S_U$.

Definition 15. An algebraic object will be called *particular* if it is generated by one its point. A particular object is called *transitive* if it is generated by any its point.

Definition 16. A subset U is said to be a *generator* of the object if it generates this object and any proper subset of U does not have this property.

Proposition 5. *An algebraic object is transitive if and only if it does not have a proper subobject.*

In fact, if the object Ω is transitive and Ω' is its subobject, then for any $x \in \Omega'$ the subobject Ω_x generated by x is the whole Ω . On the other hand it holds $\Omega_x \subset \Omega'$, thus it must be $\Omega' = \Omega$.

Conversely, if any subobject of Ω is equal to Ω , then it holds also for any generated object Ω_x ($x \in \Omega$), which means the transitivity of Ω .

It is easily seen that any transitive algebraic object has the following *connection property*: for any two elements x, x' there exists an operator a such that $x' = ax$. This property is equivalent with the transitivity property.

4. The stationary substructure

For a given operator $\alpha \in F$, let D_α denote the domain of this operator, i.e. the set of all the elements $x \in X$ such that product αx is defined.

Definition 17. The set H_U of all the operators α leaving a given subset $U \subset X$ invariant, i.e. satisfying the following condition

$$(8) \quad D_\alpha \cap U \neq \emptyset, \quad \alpha U \subset U,$$

will be called a *stabilisator* of the set U .

The stabilisator of an one-element subset $\{x\}$ we shall denote by H_x .

By the term „substructure” of F we shall mean a subsemigroupoid, subgroupoid etc. according to that which structure F represents.

Proposition 6. If a subset $U \subset X$ satisfies the condition: if $\alpha \in F$ then

$$(9) \quad \alpha U \subset U \text{ or } (\alpha U) \cap U = \emptyset,$$

then H_U is a substructure of F .

Proof. (In the case when F is a groupoid.) It follows from (9) and from the associative condition (C) that if product $\alpha\beta$ is defined and $\alpha, \beta \in H_U$, then $D_{\alpha\beta} \cap U \neq \emptyset$ and $(\alpha\beta)U = \alpha(\beta U) \subset \alpha U \subset U$, thus the product $\alpha\beta \in H_U$.

Similarly, if $\alpha \in H_U$, then $D_{\varepsilon_\alpha} \cap U \neq \emptyset$ and $D_{\delta_\alpha} \cap U \neq \emptyset$ and from the neutrality condition we have $\varepsilon_\alpha U \subset U$ and $\delta_\alpha U \subset U$, thus $\varepsilon_\alpha, \delta_\alpha \in H_U$.

Now we want to prove that if $\alpha \in H_U$ then also $\alpha^{-1} \in H_U$. Assume $\alpha \in H_U$. It holds $D_{\alpha^{-1}} \cap U \neq \emptyset$ and we have

$$\alpha^{-1}(\alpha U) = (\alpha^{-1}\alpha)U = \varepsilon_\alpha U \subset U.$$

On the other hand the inclusion $\alpha U \subset U$ implies $\alpha^{-1}(\alpha U) \subset \alpha^{-1}U$. Thus we have $(\alpha^{-1}U) \cap U \neq \emptyset$. From the condition (9) we get immediately $\alpha^{-1}U \subset U$, thus $\alpha^{-1} \in H_U$, which completes the proof.

Since a one-element subset $\{x\}$ satisfies condition (9) trivially, so the following statement holds:

Corollary 1. Every stabilisator H_x is a substructure of F .

The condition (9) was needed to prove that $\alpha^{-1} \in H_U$, which is not necessary in the case where F is a semigroupoid or a semigroup. Thus we can formulate the following

Corollary 2. If F is a semigroupoid or a semigroup then any stabilisator H_U is a substructure of F .

5. Invariant decompositions

We say family $W = (X_\lambda)_{\lambda \in L}$ of subsets of a set X to form a *decomposition* of this set if W contains at least two different subsets of X and moreover it holds

$$X = \bigcup_{\lambda \in L} X_\lambda \text{ and } X_\lambda \cap X_\mu = \emptyset \text{ for } \lambda \neq \mu.$$

