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Algebraic Objects

Introduction. In the present paper we define the notion of an algebraic object as some type of the algebraic operator structure. Then we give a series of definitions bounded with this notion and we prove some propositions and theorems about the algebraic objects. From the algebraic point of view this paper may be regarded as a treatise about such operator structures which have only exterior operations.

The standpoint to definition of algebraic object was the notion of geometric object (cf. [1], [4]), and more exactly the notion of so-called „abstract” geometric object (cf. [4], p. 24). The acting of a transformation group G on the fibre X of this object may be treated as an exterior operation on X with G as the set of operators. In the paper we consider operator structures which are more general than these one which we obtain from geometric objects.

Moreover, algebraic objects may serve as algebraic models of abstract automatic machines (cf. [3]). We shall deal with this problem more exactly in a separate paper.

We take all the fundamental terms, notions and theorems of the theory of algebraic structures used in this paper after Bourbaki [2].

I. The notion of algebraic object

1. Let A be an arbitrary set. We denote by (A, g_i, f_j, F_j) $i = 1, \dots; j = 1, \dots$ the algebraic structure defined on A by interior operations g_i

$$g_i : A^2 \ni (a, b) \rightarrow g_i(a, b) \in A$$

and exterior operations f_j

$$f_j : F_j \times A \ni (a, a) \rightarrow f_j(a, a) \in A$$

with the operator sets F_j . The results of the above operations we shall briefly write multiplicatively: $g(a, b) = ab$ and $f(a, a) = aa$ and we shall call them in both the cases the „products”.

Definition 1. By a *pure operator structure* we shall mean an algebraic structure which have only exterior operations.

Definition 2. A multiplicative system F with one interior operation defined for some ordered pairs $(\alpha, \beta) \in F \times F$ will be called a *semi groupoid* if this operation satisfies the following axioms:

(a) If in the equation

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma$$

on one of its sides or both the products $\beta\gamma$ and $\alpha\beta$ are defined, then both sides of the equation are defined and the equality holds.

(b) To every element a of F there exists exactly one left unit ε_a and exactly one right unit δ_a such that

$$\varepsilon_a a = a \delta_a = a.$$

(c) If the product $\alpha\beta$ is defined then $\delta_\alpha = \varepsilon_\beta$.

Definition 3. A semigroupoid will be called a *groupoid* * if, in addition to axioms (a)-(c), also the following condition holds

(d) To every element a there exists exactly one element a^{-1} (inverse to a) such that

$$aa^{-1} = \varepsilon_a, \quad a^{-1}a = \delta_a.$$

We omit the simple proof of the following proposition.

Proposition 1. If the multiplication $(\alpha, \beta) \rightarrow \alpha\beta$ is defined on the whole Cartesian product $F \times F$ then the semigroupoid is a semigroup with the unit-element and the groupoid is a group.

1. The fundamental notions

Definition 4. A pure operator algebraic structure $\Omega = (X, F)$ ** with the basic set X and the set of operators F will be called a *left algebraic object* over F if the exterior multiplication $(\alpha, x) \rightarrow \alpha x$ defined for some pairs $(\alpha, x) \in F \times X$ satisfies the following axioms:

(A) F has a structure at least of a semigroupoid.

(B) The exterior multiplication is associative in the following sense: if the products αx and $\alpha\beta$ are defined then both sides of the equation

$$\alpha(\beta x) = (\alpha\beta)x$$

are defined and this equality holds.

(C) Any unit is a neutral operator, i.e. if αx is defined then $\varepsilon_\alpha x$ and $\delta_\alpha x$ are defined and the following equalities hold

$$\varepsilon_\alpha x = \delta_\alpha x = x.$$

* There exist also another definitions of groupoid, cf. [6].

** In the following we shall omit the sign of exterior multiplication f in the notation (X, F, f) (as in section 1).