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Invariant Functionals on Locally Compact Topological Spaces

1. Introduction

The purpose of this paper is to state a theorem concerning the existence of invariant linear functionals on locally compact topological spaces. This theorem is peculiarly simple in the case of compact space and gives, as corollaries, a generalization of Krylov-Bogolioubov theorem on the existence of invariant measures on compact topological spaces and the existence of Haar integral on compact Abelian topological groups.

For a detailed account of invariant functionals and measures, we refer the reader to [2] and [3].

Let X be a locally compact Hausdorff topological space. Let $E = C_{00}(X)$ denote the set of all real-valued continuous functions φ on X which vanish outside of a compact sets, i.e. such that there exists a compact subset F of X (depending upon φ) such that $\varphi(x) = 0$ for all $x \in X \setminus F$. For $\varphi \in E$, let $\|\varphi\| = \sup \{|\varphi(x)| : x \in X\}$. It is obvious that E is a real normed linear space.

Let E^* denote the real topological vector space of all linear functionals on E with pointwise convergence topology. Let $E_+ = \{\varphi \in E : \varphi > 0\}$ and $E_+^* = \{u \in E^* : u(\varphi) > 0 \text{ for all } \varphi \in E_+\}$.

Let \mathcal{T} denote a family of distinct homeomorphisms of X onto itself. A functional $u \in E^*$ such that $u(\varphi) = u(\varphi T)$ for all $\varphi \in E$ and $T \in \mathcal{T}$, is said to be *invariant under the family \mathcal{T}* .

The main theorem of this paper is based on the following well known Markov-Kakutani fixed-point theorem ([1], chapitre II, appendice).

Let E be a real Hausdorff topological vector space, K — a non-empty, compact and convex subset of E . If $(U_i)_{i \in I}$ is a family of linear transformations of E into itself, such that for all $i, \alpha \in I$ $U_i U_\alpha = U_\alpha U_i$, $U_i(K) \subset K$ and restrictions of U_i to K are continuous, then there exists a point $x \in K$ such that $U_i(x) = x$ for each $i \in I$.

2. Invariant functionals

We shall first prove the following

Theorem 1. *Let $\varphi \in E_+$, $\|\varphi\| > 1$ and suppose that:*

- (i) *identical mapping belongs to \mathcal{G} ,*
 - (ii) *if $T \in \mathcal{G}$, then $T^{-1} \in \mathcal{G}$,*
 - (iii) *for each $S, T \in \mathcal{G}$, $ST = TS$,*
 - (iv) *for each $x, y \in X$, there is $T \in \mathcal{G}$ such that $y = T(x)$,*
 - (v) *if $\lambda_1\varphi T_1 + \dots + \lambda_n\varphi T_n \geq 0$ and $T_i \in \mathcal{G}$, then $\lambda_1 + \dots + \lambda_n \geq 0$.*
- Then there exists a functional $u \in E_+^*$, invariant under \mathcal{G} and such that $u(\varphi) = 1$.*

Proof. The set

$$K = \{u \in E_+^* : u(\varphi T) = 1 \text{ for each } T \in \mathcal{G}\}$$

is obviously closed and convex. We shall prove that it is non-empty and compact.

To this end, let V denote the subspace of E generated by the set $\{\varphi T : T \in \mathcal{G}\}$. From (v) it follows that the functional $v: V \rightarrow \mathbb{R}$ defined by

$$v: \sum_{i=1}^n \lambda_i \varphi T_i \rightarrow \sum_{i=1}^n \lambda_i$$

is linear. v can be extended to a linear functional $\tilde{v} \in E_+^*$. It is obvious that $\tilde{v} \in K$.

We shall now show that

$$K \subset L = \prod_{\varphi \in E} [-k_\varphi \|\varphi\|, k_\varphi \|\varphi\|],$$

where k_φ are positive integers suitably chosen. Since K is closed and L is compact by Tikhonov theorem, this will imply that K is compact, too.

Let $M = \varphi^{-1}((1, +\infty))$ and $\psi \in E$. Since (iv) holds, $(T(M))_{T \in \mathcal{G}}$ is an open covering of $\text{supp } \psi$. By compactness, there is a finite number of sets $T_1(M), \dots, T_{k_\varphi}(M)$ which cover $\text{supp } \psi$. It is then easy to see that

$$\eta = \sum_{i=1}^{k_\varphi} \varphi T_i^{-1}$$

is a function from E_+ and

$$|\psi| \leq \|\psi\| \cdot \eta$$

because $\eta(x) > 1$ for all $x \in \text{supp } \psi$. This implies that for all $u \in K$

$$|u(\psi)| \leq u(|\psi|) \leq \|\psi\| \sum_{i=1}^{k_\varphi} u(\varphi T_i^{-1}) = k_\varphi \|\psi\|$$

and therefore $u \in L$.

It is obvious that for every $u \in E^*$ and $T \in \mathfrak{G}$ the functional $u_T: E \rightarrow R$ defined by the formula $u_T(\varphi) = u(\varphi T)$ is linear and that

$$U_T: u \rightarrow u_T$$

is a linear and continuous mapping of E^* into itself. Condition (iii) implies that $U_S U_T = U_T U_S$ for each $S, T \in \mathfrak{G}$, and the inclusions $U_T(K) \subset K$ follow immediately from the definitions of K and U_T . Thus, from Markov-Kakutani theorem it follows that there exists a functional $u \in K$, such that $U_T(u) = u$ and therefore $u_T = u$ for all $T \in \mathfrak{G}$.

3. Invariant functionals on compact topological spaces

If X is compact we can omit all conditions of theorem 1 except (iii). More precisely, in this case we have the following

Theorem 2. *If X is a compact, Hausdorff topological space, and \mathfrak{G} is a family of homeomorphisms of X onto itself, such that $ST = TS$ for every $S, T \in \mathfrak{G}$, then there exists a functional $u \in E_+^*$, invariant under \mathfrak{G} and such that $u(1) = 1$.*

It is easily seen that the set

$$K = \{u \in E_+^* : u(1) = 1\}$$

is non-empty, convex and compact in E . The mappings U_T satisfy the assumptions of Markov-Kakutani theorem (the detailed arguments are very like those of the proof of theorem 1 and will be left to the reader), so that the assertion of theorem 2 holds true.

4. Krylov — Bogolioubov theorem

Let X be a compact Hausdorff topological space, and let G be an arbitrary Abelian group. Suppose that $(T_a)_{a \in G}$ is a family of homeomorphisms of X onto X such that

$$T_a T_b = T_{a+b}$$

for all $a, b \in G$. The pair $(X, (T_a)_{a \in G})$ is called a *compact dynamical system*.

A Borel measure μ on X such that $\mu(A) = \mu(T_a(A))$ for all $a \in G$ and all μ -measurable sets $A \subset X$, is said to be *invariant under family $(T_a)_{a \in G}$* , or *invariant measure* on compact dynamical system. If in addition $\mu(X) = 1$, then μ is called a *normed invariant measure*. It is well known that the existence of a Borel measure invariant under family $(T_a)_{a \in G}$ is equivalent to the existence of a positive functional on Banach space of all real-valued continuous functions on X , invariant under the same family. As an immediate consequence of theorem 2 we obtain the following generalization of the classical result of Krylov and Bogolioubov ([4]).

Theorem 3. *For every compact dynamical system $(X, (T_a)_{a \in G})$ there exists a positive linear functional on E invariant under family $(T_a)_{a \in G}$. Therefore there exists a Borel invariant normed measure on X .*

In the special case of the additive group R of real numbers, from theorem 3 we obtain the existence of invariant measures on compact continuous flows $(X, (T_r)_{r \in R})$.

Similarly, if T is a homeomorphism of X onto X and $T_n = T^n$ for every integer n , then theorem 3 implies the existence of invariant measures on compact discrete flows (X, T) . Finally, if G is an arbitrary topological Abelian compact group and T_a denotes, for every $a \in G$, the translation $x \rightarrow a + x$, then theorem 3 yields the existence of the Haar integral on G .

REFERENCES

- [1] N. Bourbaki, *Espaces vectoriels topologiques*, Hermann, Paris 1953.
- [2] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Springer-Verlag, Berlin—Göttingen—Heidelberg 1963, 184-260.
- [3] K. Jacobs, *Neuere Methoden und Ergebnisse der Ergodentheorie*, Springer-Verlag, Berlin—Göttingen—Heidelberg 1960, 93-100.
- [4] J. C. Oxtoby, *Ergodic sets*, Bull. Amer. Math. Soc. 58 (1952), 116-136.