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Non-existence of Certain Concomitants of the Singular Tensor Λ^a_{ij} and its Geometrical Interpretation

1. Let S_n be a surface, imbedded in an N-dimensional space L_N with affine connection. S_n can be defined by a system of Pfaff forms ω^J, ω^J_K $(I, K = 1, \ldots, N)$ which satisfy the following structure equations

$$D\omega^J = \omega^L \wedge \omega^J_L + R^J_{PQ}\,\omega^P \wedge \omega^Q\,,$$
$$D\omega^J_K = \omega^L_K \wedge \omega^J_L + R^J_{KPQ}\,\omega^P \wedge \omega^Q\,,$$

where R^J_{PQ}, R^J_{KPQ} denote the torsion and curvature tensors of L_N, respectively. If the first n vectors $e_1, \ldots, e_N$ of the mobile frame $m, e_1, \ldots, e_N$ of L_n are situated in the tangent space of S_n, then S_n may be defined by the differential equations

$$\omega^a = 0\,, \qquad (a = n+1, \ldots, N)\,.$$

(cf. [3]). The first prolongation of these equations by using of the Cartan' lemma yields us

$$\omega^a_i = \Lambda^a_{ij}\,\omega^j\,, \qquad (i, j = 1, \ldots, n)\,,$$

where $\Lambda^a_{ij} - \Lambda^a_{ji} = R^a_{ij}$. The functions

(1) $$\Lambda^a_{ij}$$

represent a tensor field on S_n. Λ^a_{ij} is sometime called the asymptotic tensor of surface. It represents (following Laptev [3]) the fundamental object of order 2 of S_n. This means that every geometric object, whose components depend on the derivatives up to the degree 2 of the parametrical representation of S_n, is a function (concomitant) of Λ^a_{ij}. Therefore the investigation of concomitants of this tensor is important for the geometry of surface.

In the present paper we shall investigate the possibility of construction such objects as vectors, densities, scalars and tensors which depend on Λ^a_{ij}, in a case where the tensor (1) is singular. We give only the negative results. Moreover we prove some geometrical properties of the surface which has a singular asymptotic tensor.

2. We consider the general non-symmetric case Λ^a_{ij} (that means for S_n that its torsion tensor R^a_{ij} does not vanish). In the sequel we put $a = 1, \ldots, p$.

Definition 1. The number r such that $n-r$ is equal to the dimension of a linear subspace $V \subset V^n$ spanned on all the vectors $v = (v^i)$ for which holds

$$\Lambda^a_{ik} v^k = \Lambda^a_{kj} v^k = 0 \tag{2}$$

for all $a = 1, \ldots, p$; $i, j = 1, \ldots, n$,*, will be called a horizontal rang of Λ^a_{ij}.

Definition 2. The greatest number τ for which the tensor

$$\Lambda^{[a_1}_{i_1j_1} \Lambda^{a_2}_{i_2j_2} \ldots \Lambda^{a_\tau]}_{i_\tau j_\tau} \tag{3}$$

does not vanish is called the vertical rang of Λ^a_{ij}.

Lemma 1. *The vertical rang τ is equal to the number of linear independent vectors*

$$e_{ij} \overset{df}{=} \Lambda^a_{ij} e_a\,, \quad (i, j = 1, \ldots, n)\,, \tag{4}$$

where $e_a = (\delta^\beta_a)$ $p = 1, \ldots, p$ are the unit vectors.

Proof. We obtain our statement immediately from the relations

$$e_{i_1j_1} \wedge e_{i_2j_2} \wedge \ldots \wedge e_{i_\tau j_\tau} = \tau!\, \Lambda^{[a_1}_{i_1j_1} \Lambda^{a_2}_{i_2j_2} \ldots \Lambda^{a_\tau]}_{i_\tau j_\tau} e_{a_1} \wedge e_{a_2} \wedge \ldots \wedge e_{a_\tau} \tag{5}$$

for all sequences of pairs (i_k, j_k). It means that the vectors $e_{i_1j_1}, \ldots, e_{i_\tau j_\tau}$ are independent if and only if not all the coefficients (3) in equalities (5) vanish.

Lemma 2. *The number $p-\tau$ is equal to the maximal number of linear independent vectors $v = (v_a)$ such that*

$$\Lambda^a_{ij} v_a = 0\,, \tag{6}$$

for all $i, j = 1, \ldots, n$.

Proof. The rows of the following $n^2 \times p$-matrix

$$\begin{bmatrix} \Lambda^a_{11} \\ \Lambda^a_{12} \\ \vdots \\ \Lambda^a_{nn} \end{bmatrix} \tag{7}$$

consist of the components of vectors (6). In virtue of the lemma 1 exactly τ among these vectors are linearly independent; the same holds also for the rows of matrix (8) and consequently for the columns of this matrix too. That implies the existence of $p-\tau$ vectors (v_a) such that the relations (7) hold, which was to be proved.

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* If tensor Λ^a_{ij} is symmetric, i.e. $\Lambda^a_{ji} = \Lambda^a_{ij}$, then the conditions (2) take the simpler form $\Lambda^a_{ik} v^k = 0$.

Definition 3. The tensor Λ^a_{ij} ($a = 1, \ldots, p$; $i, j = 1, \ldots, n$) will be called regular if its horizontal and vertical ranges are maximal, i.e. $r = n$ and $\tau = p$, otherwise it will be called: horizontal singular if $r < n$, verticaly singular if $\tau < p$ and singular in the general case.

§ 3. The transformation formula * of a geometric object $\Omega = (\Omega^\lambda)$ $\lambda = 1, \ldots, m$ may be given by a system of differential equations

$$(9) \qquad d\Omega^\lambda = \Phi^\lambda_K(\Omega)\,\omega^K ,$$

where the ω^K denote the invariant forms of Lie transformation group of Ω, (cf. [2], p. 327). The forms ω^K are also the elements of Lie algebra H of this group.

Let us denote by H_{Ω_0} a Lie subalgebra of H consisting of all ω^K such that

$$(10) \qquad \Phi^\lambda_K(\Omega_0)\,\omega^K = 0 , \qquad \lambda = 1, \ldots, m ,$$

i.e. H_{Ω_0} is a subalgebra corresponding to the stationary subgroup of Ω_0 (its elements leave Ω_0 invariant).

The following statement is well known: An object θ is a concomitant of the object Ω, i.e. $\theta = F(\Omega)$, if and only if every stationary subgroup H_Ω is a subgroup of the corresponding stationary subgroup $H_{F(\Omega)}$ (cf. [4]).

From this we get immediately the following

Lemma 4. *An object θ is a concomitant of the object Ω if and only if any subalgebra H_Ω is contained in the corresponding subalgebra $H_{F(\Omega)}$, where $\theta = F(\Omega)$.*

In our case, for the tensor Λ^a_{ij} the equations (9) take the form

$$(11) \qquad d\Lambda^a_{ij} = \Lambda^a_{ik}\omega^k_j + \Lambda^a_{kj}\,\omega^k_i - \Lambda^\beta_{ij}\,\omega^a_\beta ,$$

where (ω^i_j, ω^a) — the invariant forms of the direct product $GL(n) \times GL(p)$ (transformation group of tensor (1)). The corresponding invariance equations (10) have the form

$$(12) \qquad \Lambda^a_{ik}\,\omega^k_j + \Lambda^a_{kj}\,\omega^k_i - \Lambda^\beta_{ij}\,\omega^a_\beta = 0$$

for all i, j, a. These equations are satisfied for any Λ^a_{ij}, for instance, by the forms

$$(13) \qquad \omega^i_j = \delta^i_j , \qquad \omega^a_\beta = 2\delta^a_\beta .$$

It follows from the lemma 4 that every stationary subalgebra of any concomitant of (1) must contain the forms (13). In virtue of this we get easily the following:

Theorem 1. *No tensor*

$$(14) \qquad a^{i_1, \ldots, i_p, a_1 \ldots a_s}_{j_1, \ldots, j_q, \beta_1 \ldots \beta_t}$$

for which $(p-q) + 2(s-t) \neq 0$ may be constructed from the tensor (1).

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* If it is analytic.

Proof. In fact, the invariance equations for (14) are following:

$$\sum_{k=1}^{P} a^{i_1 \ldots m \ldots i_p, \ldots} \omega_m^{i_k} - \sum_{k=1}^{q} a^{\ldots, \ldots}_{j_1 \ldots m \ldots j_q, \ldots} \omega_{j_k}^{m} + \\ + \sum_{k=1}^{s} a^{\ldots, \alpha_1 \ldots \varrho \ldots \alpha_s}_{\ldots, \ldots} \omega_\varrho^{\alpha_k} - \sum_{k=1}^{t} a^{\ldots, \ldots}_{\ldots, \beta_1 \ldots \varrho \ldots \beta_t} \omega_{\beta_k}^{\varrho} = 0 . \tag{15}$$

The forms (13) must satisfy these equations for any components (14). Substituting (13) in (15) we obtain

$$a^{i_1, \ldots, \ldots \alpha_s}_{j_1, \ldots, \ldots \beta_t} [p - q + 2(s - t)] = 0$$

for all indices i, j, α, β. This completes the proof of the theorem.

Corollary 1. *No vector may be constructed from* (1).

Now, let $I = I(\Lambda_{ij}^a)$ be a density concomitant of (1). The differential equations of any density have the form

$$d \ln I = \lambda \omega_i^i + \mu \omega_a^a ,$$

where λ, μ — the weights of I related to groups $GL(n)$ and $GL(p)$ (cf. [3]).

Substituting (13) to the equation

$$\lambda \omega_i^i + \mu \omega_a^a = 0 , \tag{16}$$

we obtain $\lambda n + 2\mu p = 0$. Hence, for a corresponding power of I, we can take $\lambda = 2p$ and $\mu = -n$. Substituting it in (16) we have

$$2p \omega_i^i - n \omega_a^a = 0 . \tag{17}$$

§ 4. In the sequel we study the case where the tensor (1) is singular. Assume $r < n$. Taking as the first $n-r$ columns of a transformation matrix $A \in GL(n)$ the linear independent vectors fulfilling the relations (3) (see lemma 1) we have

$$\Lambda_{i'j'}^a = \Lambda_{ij}^a A_{i'}^i A_{j'}^j = 0 \tag{18}$$

if i' or j' is $\leqslant n-r$, i.e. the first $n-r$ rows and columns in any matrix $[\Lambda_{ij}^a]$ (a fixed) consist of zeros.

Similarly in the case $\tau < p$, taking as the first $p-r$ columns of transformation matrix $B \in GL(p)$ the linear independent vectors such that the relations (7) hold we obtain

$$\Lambda_{ij}^{a'} = \Lambda_{ij}^a B_a^{a'} = 0 \tag{19}$$

for all $a' = 1, \ldots, p-r$ and $i, j = 1, \ldots, n$.

In the following we consider the singular tensor (1) in a coordinate system $(H)_0$ for which it holds the relations (18) or (19). It is easy to verify that the stationary subalgebra of any point (Λ_{ij}^a), for which (18) or (19) holds, contains the subspaces consisting of forms

$$\omega_j^{\hat{i}} \ (\hat{i} \leqslant n-r, \ j = 1, \ldots, n) \text{ arbitrary and other } \omega_j^i = 0 , \tag{20}$$

$$\omega_{\hat{\beta}}^a (\hat{\beta} \leqslant p - \tau , \ a = 1, \ldots, p) \text{ arbitrary and other } \omega_\beta^a = 0 , \tag{21}$$

respectively. By using of lemma 3 we prove our next theorems.

Theorem 2. *No non-trivial density I may be a function of a singular tensor* (1).

Proof. Substituting (20) or (21) in (17) we get $2p\omega_{\hat{i}}^{\hat{i}}=0$ or $n\omega_{\hat{a}}^{\hat{a}}=0$ from which follows $p=0$, or $n=0$, a contradiction. According to lemma 3 the density I can not be a function of a singular tensor Λ_{ij}^{a}.

Theorem 3. *No tensor* (14) *for which* $p>0$ *and* $q=0$ *may be a concomitant of a horizontally singular tensor* (1).

Proof. If we substitute (20) into (15) and we put $\omega_j^{\hat{i}}=0$ for $\hat{i}>1$ then we obtain:

In the case where $i_k=1$ and other indices i are >1 it holds

$$a_{\dots}^{i_1..\overset{i_{k_1}}{m}..i}{}_{p}^{\dots}=0 \quad \text{for} \quad m=1,\dots,n \tag{22}$$

($\overset{i_k}{m}$ denotes that m stands in place of i_k).

In the case where $i_{k_1}, i_{k_2}=1$ and other i are >1 we have

$$a_{\dots}^{i_1..\overset{i_{k_1}}{m}..i_{k_2}..i}{}_{p}^{\dots}+a^{i_1..i_{k_1}..\overset{i_{k_2}}{m}..i}{}_{p}^{\dots}=0 \quad \text{for} \quad m=1,\dots,n\,.$$

Hence, after substituting $m=1$ we get

$$a_{\dots}^{i_1..\overset{i_{k_1}}{1}..\overset{i_{k_2}}{1}..i}{}_{p}^{\dots}=0\,.$$

It follows from this and from (22) that $a_{\beta_1\dots\beta_r}^{i_1\dots i_p,a_1\dots a_s}=0$ if at most two indices i are equal to 1. Assuming $i_{k_1}, i_{k_2}, i_{k_3}=1$ we get similarly this statement if at most the indices are equal to 1, etc. The last case where at most all p indices i are equal to 1 is general, what completes proof of the theorem.

Corollary 2. *No tensor of the form* Λ_a^{ij} *can be constructed from the horizontally singular tensor* Λ_{ij}^{a}.

Theorem 4. *If the tensor* (1) *is horizontally singular then there exists no non-vanishing scalar differentiable function* $\sigma(\Lambda_{ij}^{a})$ (i.e. *a scalar concomitant of* Λ_{ij}^{a}).

In fact, if σ is a scalar function of Λ_{ij}^{k} then the functions

$$\Lambda_a^{ij} \overset{df}{=} \frac{\partial\sigma}{\partial\Lambda_{ij}^{a}} \tag{23}$$

represent a tensor. According to the corollary 2 the tensor concomitant (23) does vanish, thus $\sigma=\text{const}$, which proves the theorem.

In a case where the tensor (1) is vertically singular we obtain by a similar manner the following

Theorem 5. *No tensor* (14) *with* $t>0$ *and* $s=0$ *may be a concomitant of a vertically singular tensor* (1).

The corollary 2 and the theorem 4 hold in this case too.

The geometrical interpretation

The infinitesimal displacement of the mobile frame $m, e_1, \ldots, e_N$ of L_N is defined by the relations

$$dm = \omega^I e_I \,, \quad de_I = \omega_I^K e_K \qquad (I, K = 1, \ldots, N)\,.$$

Hence, after putting $\omega^a = 0$, $\omega_i^a = \Lambda_{ij}^a \omega^i$ (cf. section 1) we get the following formulae for a displacement of the tangent frame $m, e_1, \ldots, e_n$ on S_n:

$$dm = \omega^i e_i \tag{24}$$

$$de_i = \omega_i^k e_K + \Lambda_{ik}^a \omega^k e_a$$

$(i, j, k = 1, \ldots, n)$. We have from (24)

$$d^2m = d(\omega^i e_i) = (d\omega^i + \omega^j \omega_j^i) e_i + \Lambda_{ij}^a \omega^i \omega^j e_a \,.$$

After substituting here $e_{ij} = \Lambda_{ij}^a e_a$ we get

$$d^2m = (d\omega^i + \omega^j \omega_j^i) e_i + \omega^i \omega^j e_{ij} \,.$$

The linear space spanned on the vectors $\{e_i, e_{ij}\}$ is called the (first) osculating plane of S_n in a point m. It is easy to verify that it is invariant under the displacement (24).

Definition 4. A surface S_n is called space-degenerated of rang τ if the dimension of its osculating plane is not maximal and equal $n+\tau < n+$ $+\min(n^2, N-n)$.

It follows immediately from the lemma 1 that the following holds.

Theorem 6. *A surface is space degenerated of rang τ if and only if its asymptotic tensor field* (1) *is vertically singular of rang τ in each point.*

Definition 5. A surface is called tangentially degenerated of rang $r < n$ if its tangent plane E_n depends on r parameters and remains immovable by the displacement of point of surface along a plane E_{n-r}, contained in this surface (S_n represents then a r-parametric family of planes E_{n-r}, cf. [1]). We shall prove our next theorem in the case where the space L_N is a flat affine space A_N (i.e. where $R_{PQ}^J = R_{KPQ}^J = 0$). The structure equations of A_N are of the form

$$D\omega^J = \omega^L \wedge \omega_L^J \,, \qquad D\omega_K^J = \omega_K^L \wedge \omega_L^J \,. \tag{25}$$

Theorem 7. *A surface S_n imbedded in an affine space A_N is tangentially degenerated of rang $r < n$ if and only if its asymptotic tensor field* (1) *is horizontally singular of rang r in each point.*

Proof of sufficiency. We choose the tangent frame $m, e_1, \ldots, e_n$ in such a manner that the tensor Λ_{ij}^a satisfies the relations (18) in each point. In the following we assume that the indices a, b run over $1, \ldots, n-r$; $i, j, k, m = 1, \ldots, n$ and $p, q = n-r+1, \ldots, n$. Then we have in view of (18)

$$\Lambda_{ia}^a = \Lambda_{aj}^a = 0 \,. \tag{27}$$

We state that through each point m passes an $n-r$ dimensional plane E_{n-r} spanned on the first $n-r$ tangent vectors $e_1, \ldots, e_{n-r}$ and lying in S_n.

The tensor field (1) satisfies the following differential equations (cf. [3])

(28) $$d\Lambda^{\alpha}_{ij} = \Lambda^{\alpha}_{im}\omega^{m}_{j} + \Lambda^{\alpha}_{mj}\omega^{m}_{i} - \Lambda^{\beta}_{ij}\omega^{\alpha}_{\beta} + \Lambda^{\alpha}_{ijk}\omega^{k},$$

where ω^k are basic forms.

After putting (27) in (28) we obtain $\Lambda^{\alpha}_{abk}\omega^k = 0$ and hence, while the forms ω^k are independent, we get

(29) $$\Lambda^{\alpha}_{abk} = 0 .$$

Similarly, substituting in (28) the indices a, q in place of i, j we get

(30) $$\Lambda^{\alpha}_{pq}\omega^{p}_{a} + \Lambda^{\alpha}_{aqk}\omega^{k} = 0 .$$

In view of (29) and of symmetry of coefficients Λ^{α}_{ijk} we have $\Lambda^{\alpha}_{aqb} = 0$ and therefore, taking $\omega^p = 0$, we get from (30)

(31) $$\Lambda^{\alpha}_{pq}\omega^{p}_{a} = 0 \quad (\mathrm{mod}\,\omega^p) .$$

Let us put

(32) $$v^j = \begin{cases} 0 \text{ for } j = 1, \ldots, n-r \\ \omega^{j}_{a} \text{ for } j = n-r, \ldots, n \end{cases}$$

by a fixed a. Then the relations (31) can be written in the form $\Lambda^{\alpha}_{jq}v^j = 0$ and from symmetry of Λ^{α}_{ij} also $\Lambda^{\alpha}_{qj}v^j = 0$. In view of (27) and of the assumption that the horizontal rang of Λ^{α}_{ij} is equal exactly to r, the subspace V_0 (cf. definition 1) of all vectors satisfying (2) is here spanned on vectors $e_1, \ldots, e_{n-r}$, i.e. $V_0 = E_{n-r}$. Thus it must hold $(v^j) \in E_{n-r}$ and hence in view of (32) $v^j = 0$.

Thus we have finally according to (31) and (32)

(33) $$\omega^{p}_{a} = 0 \ (\mathrm{mod}\,\omega^p) \cdot$$

We have from (25)

(34) $$d\omega^{p}_{a} = \omega^{b}_{a}\wedge\omega^{p}_{b} + \omega^{q}_{a}\wedge\omega^{p}_{q} ,$$

while $\omega^a = 0$ on S_n. In view of (33) and (34) we obtain $d\omega^{p}_{a} = 0$ $(\mathrm{mod}\,\omega^p)$, which means that the system of differential equations $\omega^p = 0$ is completely integrable. Thus these equations define a family of subsurfaces which are tangent to E_{n-r} in each point. From (24), (27) and (33) we get the following relations

(35) $$\left.\begin{aligned} dm &= \omega^{a}e_a \\ de_a &= \omega^{b}_{a}e_b \end{aligned}\right\} \quad (\mathrm{mod}\,\omega^p) .$$

That means, that these subspaces are the plans of dimension $n-r$. Thus they are identical with the plans E_{n-r}. By an infinitesimal displacement of a tangent vector v in any direction u we have

$$dv = (dv^i e_i) = (dv^i + \omega^{i}_{j}v^j)e_i + \Lambda^{\alpha}_{ij}v^i u^j e_\alpha .$$

But it holds $\Lambda^{a}_{ij}u^{i}=0$ for $u\in E_{n-r}$. Hence $\Lambda^{a}_{ij}v^{i}u^{j}e_{a}=0$, which means that the vector v remains in a tangent plane. Since v is arbitrary tangent vector, also the whole tangent plane E_n is immovable by a displacement on E_{n-r}, q.e.d.

In order to prove the necessity it is sufficient to remark that the relations (35) imply the relations (2) for any vector v from a plane E_{n-r} (we get that by the comparison (35) with (24)). Thus it must be (we assume that S_n is tangentially degenerated of maximal rang r) $V_0=E_{n-r}$, which means that the asymptotic tensor (1) is horizontally singular of rang r in each point. This completes the proof of the theorem.

The theorems of the section 3 have immediately simple meaning for the surface. They say about non-existence of certain geometric objects fields of order 2, invariantly bounded with the surface. For example, the following holds.

Theorem 8. *On a singular surface do not exist the fields of relative invariants (densities) and fields of differentiable absolute invariants (scalars) of order 2.*

From the corollary 1 we get the following

Theorem 9. *On any surface imbedded in L_N (or in A_N) do not exist the fields of vectors of order 2.*

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