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A Generalization of the Second Theorem of O. Hanner

In his paper [3] Hanner gives the proofs of two theorems, called respectively first and second theorem of Hanner, concerning the notion of the $ANR(\mathfrak{M})$-spaces.

First theorem of Hanner. *Every open subset of an $ANR(\mathfrak{M})$-space is an $ANR(\mathfrak{M})$-space.*

Second theorem of Hanner. *If a metrizable space X is the countable union of open sets G_i $(i = 1, 2 \ldots)$ which are $ANR(\mathfrak{M})$-spaces, then X is an $ANR(\mathfrak{M})$-space.*

The definitions of the $ANR(\mathfrak{M})$-space and the proofs of the theorems of Hanner are to be found in [1] (Chapter IV, p. 85-99), where the following problem is raised: Is it true that a metrizable space X in which every point has a neighbourhood being an $ANR(\mathfrak{M})$ is necessarily an $ANR(\mathfrak{M})$?

This paper contains a positive answer to this question.

First we shall prove the following

Lemma 1. *If a metrizable space X is the union of his open pairwise disjoint subsets being the $ANR(\mathfrak{M})$-spaces, then X is an $ANR(\mathfrak{M})$-space.*

Proof. We may assume that X is a closed subset of a metric space Y. We have

$$X = \bigcup_{\nu \in M} G_\nu, \quad \nu \neq \mu \Rightarrow G_\nu \cap G_\mu = \emptyset, \quad G_\nu \in ANR(\mathfrak{M}),$$

where G_ν are open in X and M denotes an arbitrary set of indexes. Suppose first that there exists a family $\{U_\nu\}_{\nu \in M}$ such that $U_\nu \supset G_\nu$, U_ν are open in Y, and U_ν are pairwise disjoint. We have

$$G_\nu = X - \bigcup_{\nu \neq \mu} G_\mu$$

hence G_ν is a closed subset of Y. G_ν being an $ANR(\mathfrak{M})$-space, we may find a retraction $r_\nu\colon V_\nu \to G_\nu$, where V_ν is an open subset of Y. Taking

$$W_\nu = U_\nu^* \cap V_\nu, \quad i_\nu = r_\nu|_{W_\nu} \quad \text{and} \quad i = \bigcup_{\nu\in M} i_\nu$$

we have a retraction

$$i\colon \bigcup_{\nu \in M} W_\nu \to X .$$

If the family $\{G_\nu\}_{\nu\in M}$ satisfies the condition: ($\ast$) there exists a number $d > 0$ such that $\nu \neq \mu \Rightarrow \varrho(G_\nu, G_\mu) > d$, then X is an $ANR(\mathfrak{M})$ since we may take as $\{U_\nu\}_{\nu\in M}$ the sets

$$U_\nu = \bigcup_{x \in G_\nu} K(x, d/3) ,$$

$K(x, r)$ being the open ball with center x and radius r.

In the general case we take

$$G_{\nu i} = \{x \in G_\nu\colon \varrho(x, X - G_\nu) > 1/i\} \quad \nu \in M \ (i = 1, 2, \ldots) .$$

Since $G_{\nu i}$ is open in G_ν, $G_{\nu i}$ is an $ANR(\mathfrak{M})$ by the first theorem of Hanner. We can easily verify that the family $J_i = \{G_{\nu i}\}_{\nu\in M}$ satisfies the condition ($\ast$) Hence

$$\bigcup_{\nu\in M} G_{\nu i}$$

is an $ANR(\mathfrak{M})$, and

$$X = \bigcup_{i=1}^{\infty} \bigcup_{\nu \in M} G_{\nu i}$$

is an $ANR(\mathfrak{M})$ by the second theorem of Hanner. The proof of lemma 1 is thus completed.

Now let us pass to the general case. X being metric space, it is paracompact; therefore we may assume that

$$X = \bigcup_{\nu\in M} G_\nu, \quad G_\nu \text{ open in } X, \quad G_\nu \in ANR(\mathfrak{M})$$

and that $\{G_\nu\}_{\nu\in M}$ is locally finite.

We define by induction a sequence $G^k = \{G_\nu^k\}_{\nu\in M}$ $(k = 0, 1, \ldots)$ of open and locally finite coverings of the space X. For $k = 0$,

$$\mathcal{G}^0 = \{G_\nu\}_{\nu\in M} = \{G_\nu^0\}_{\nu\in M} .$$

The covering $\mathcal{G}^k = \{G_\nu^k\}_{\nu\in M}$ being defined, we define a covering $\mathcal{G}^{k+1} = \{G_\nu^{k+1}\}_{\nu\in M}$ as an open and locally finite covering of the space X satisfying the condition

$$\overline{G_\nu^{k+1}} \subset G_\nu^k .$$

That such a covering exists we may deduce from the very well known theorem on paracompact spaces ([2], p. 209).

For every covering $\mathfrak{G}^k$ and for every point $x \in X$ we define a positive integer $a_k(x)$ by the conditions: $a_k(x) = n$ if and only if

(i) There exists a sequence of n sets $G^k_{\nu_1}, \ldots, G^k_{\nu_n}$ such that for every neighbourhood V_x of x we have $G^k_{\nu_i} \cap V_x \neq \emptyset$ $(i = 1, \ldots, n)$.

(ii) There exists a neighbourhood V^k_x such that the sets $V^k_x \cap G^k_\nu$ are non empty only for $\nu = \nu_i$ $(i = 1, \ldots, n)$.

It is evident that $a_k(x)$ is well defined for every point $x \in X$ and for every covering $\{G^k_\nu\}_{\nu \in M}$.

Let

$$K^k_n = \{x \in X : a_k(x) = n\}$$

and

$$A^k_{\nu_1,\ldots,\nu_n} = \bigcap_{i=1}^{n} G^k_{\nu_i} \cap \bigcup_{x \in K^k_n} V^k_x$$

where $(\nu_1, \ldots, \nu_n) \in M^n$ and V^k_x is a neighbourhood of x satisfying the condition (ii) of the definition of $a_k(x)$. We have:

a) $A^k_{\nu_1,\ldots,\nu_n}$ is open for every $(\nu_1, \ldots, \nu_n) \in M^n$ and $k = 1, 2, \ldots$

b) $A^k_{\nu_1,\ldots,\nu_n} \subset \bigcap_{i=1}^{n} G^k_{\nu_i}$, so that $A^k_{\nu_1,\ldots,\nu_n}$ is an $ANR(\mathfrak{M})$.

c) The sets $A^k_{\nu_1,\ldots,\nu_n}$ are pairwise disjoint.

To prove c) let us suppose that $\mu_1 \neq \nu_i$ for every $i = 1, \ldots, n$ and let there exist a point $y \in X$ such that

$$y \in A^k_{\nu_1,\ldots,\nu_n} \cap A^k_{\mu_1,\ldots,\mu_n}.$$

Since $y \in A^k_{\nu_1,\ldots,\nu_n}$, there exists a point $x \in K^k_n$ such that

$$y \in V^k_x \cap \bigcap_{i=1}^{n} G^k_{\nu_i}.$$

By the definition of V^k_x, the neighbourhood V^k_x has empty intersections with all the sets G^k_ν, $\nu \neq \nu_i$. But $y \in V^k_x \cap G^k_{\mu_1}$ and $\mu_1 \neq \nu_i$. This is impossible, and this implies that the sets $A^k_{\nu_1,\ldots,\nu_n}$ are pairwise disjoint.

Let

$$A^k_n = \bigcup_{(\nu_1,\ldots,\nu_n) \in M^n} A^k_{\nu_1,\ldots,\nu_n}.$$

Lemma 1 implies that A^k_n is an $ANR(\mathfrak{M})$. By the second theorem of Hanner we have that

$$\mathfrak{B}^k = \bigcup_{n=1} A^k_n$$

is an $ANR(\mathfrak{M})$.

Now we shall prove the following

Lemma 2. *For every* $x \in X$,

$$x \notin \mathfrak{B}^k \Rightarrow a_{k+1}(x) < a_k(x).$$