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### Note on a Problem of K. Borsuk

Given any metric space  $X$  we denote by  $2^X$  the space of all non-empty compacta lying in  $X$  with the distance given by the Hausdorff formula

$$\rho(A, B) = \max[\sup_{x \in B} \rho(x, A), \sup_{x \in A} \rho(x, B)].$$

Given a positive integer  $n$  let us denote by  $X^{[n]}$  the set of all elements of  $2^X$  containing at most  $n$  points.

We say that the space  $X$  is an  $ANR(\mathfrak{M})$  if and only if  $X$  is metrizable and for each homeomorphism  $h$ , mapping  $X$  onto a closed subset  $h(X)$  of a metrizable space  $Y$ ,  $h(X)$  is a neighbourhood retract in  $Y$ .

We say that the space  $X$  is an  $ANR$  if and only if  $X$  is an  $ANR(\mathfrak{M})$  and  $X$  is compact.

In [1] (Chapter IX, p. 215) the following problem is raised: Is it true that  $X \in ANR(\mathfrak{M})$  implies that  $X^{[n]} \in ANR(\mathfrak{M})$ ?

In this paper I shall prove that under some assumptions on the space  $X$  the answer to this question is positive. Namely we have the following

**Theorem.** *If the paracompact space  $X$  has a covering  $\{A_\nu\}_{\nu \in M}$  ( $M$  denotes an arbitrary set of indexes) satisfying the following conditions:*

- (i)  $A_\nu$  are open in  $X$
- (ii)  $\bar{A}_\nu \in ANR(\mathfrak{M})$  for all  $\nu \in M$
- (iii)  $\bar{A}_\nu$  are compact for all  $\nu \in M$
- (iv)  $\bar{A}_{\nu_1} \cap \dots \cap \bar{A}_{\nu_k} \in ANR(\mathfrak{M})$  for all  $(\nu_1, \dots, \nu_k) \in M^k$

then  $X^{[n]}$  is an  $ANR(\mathfrak{M})$ .

**Proof.** The proof is based on a theorem of Ganea ([2]), p. 315) and a generalized version of a theorem of Hanner [4].

**Theorem of Ganea.** *If  $X$  is an  $ANR$  then  $X^{[n]}$  is an  $ANR$ .*

The generalized theorem of Hanner. *If every point of the metric space  $X$  has a neighbourhood being an  $ANR(\mathfrak{M})$  then  $X$  is an  $ANR(\mathfrak{M})$ .*

By the generalized theorem of Hanner we see that the space  $X$  satisfying the assumptions (i)-(iv) is an  $ANR(\mathfrak{M})$  ( $X$  is a metric space since it is paracompact and locally metrizable).  $X^{[n]}$  being a metric space, by the same theorem (i.e. the generalized theorem of Hanner) to prove that  $X^{[n]}$  is an  $ANR(\mathfrak{M})$  it is sufficient to show that every point of  $X^{[n]}$  has a neighbourhood being an  $ANR(\mathfrak{M})$ .

Let  $A$  be an arbitrary element of  $X^{[n]}$ . From the definition of  $X^{[n]}$   $A$  is a compact subset of  $X$ . Hence we may find a finite collection of sets  $A_1, \dots, A_k$  such that

$$A_i \in \{A_\nu\}_{\nu \in M} \quad \text{for} \quad 1 \leq i \leq k.$$

and

$$A \subset \bigcup_{i=1}^k A_i.$$

The closure

$$\overline{\bigcup_{i=1}^k A_i} = \bigcup_{i=1}^k \bar{A}_i$$

is a compact set (condition (iii)). Applying the known theorem on the union of closed  $ANR(\mathfrak{M})$ -spaces ([1], Chapter IV, p. 90) and assumption (iv) of our theorem we deduce that the union

$$\bigcup_{i=1}^k \bar{A}_i \in ANR(\mathfrak{M}),$$

and being a compact set it is an  $ANR$ .

By the theorem of Ganea we have

$$\left[ \bigcup_{i=1}^k \bar{A}_i \right]^{[n]} \in ANR.$$

From the evident relation

$$\left[ \bigcup_{i=1}^k A_i \right]^{[n]} \subset \left[ \bigcup_{i=1}^k \bar{A}_i \right]^{[n]} \in ANR \subset ANR(\mathfrak{M})$$

and from the first theorem of Hanner ([1], Chapter IV, p. 96) we deduce that the set

$$\left[ \bigcup_{i=1}^k A_i \right]^{[n]}$$

is an  $ANR(\mathfrak{M})$  as an open subset of an  $ANR(\mathfrak{M})$ .

But

$$\bigcup_{i=1}^k A_i$$

is open in  $X$  and

$$A \subset \bigcup_{i=1}^k A_i$$

hence the set

$$\left[ \bigcup_{i=1}^k A_i \right]^{[n]}$$

is a neighbourhood of the set  $A$  in  $X^{[n]}$ . This completes the proof.

Let us mention that the class of all spaces satisfying the assumptions of our theorem contains the class of all paracompact finite dimensional manifolds. As  $\{A_\nu\}_{\nu \in M}$  we can take the family of Euclidean balls for which the conditions (i)-(iv) can be easily verified.

#### REFERENCES

- [1] K. Borsuk, *Theory of retracts*, Warszawa 1967.
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- [3] O. Hanner, *Some theorems on absolute neighborhood retracts*, Arkiv for Mat. 1 (1951), 375—382.
- [4] E. Tutaj, *A generalization of the second theorem of O. Hanner*, Zeszyty Naukowe UJ, Prace Matematyczne, this issue, 109—112.