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On Singular Solutions of a Matrix Multiplicative Functional Equation

1. The well known Cauchy's equation for matrices

$$F(AB) = F(A)F(B)\,, \tag{1}$$

where A, B denote the $n\times n$ matrices, F is an $m\times m$ matrix-function, has in the case where A, B and F are non-singular matrices and $m \leqslant n$ the following general solutions (cf. [3])

$$F(A) = \Phi(J)CAC^{-1}, \quad J = \det A\,, \tag{2a}$$

$$F(A) = \Phi(J)\,C(A^T)^{-1}C^{-1}\,, \tag{2b}$$

$$F(A) = G(J)\,, \tag{2c}$$

C being a non-singular constant matrix, $\Phi(J)$ and $G(J)$ scalar and arbitrary matrix-functions of one scalar argument, respectively, satisfying the equation

$$G(xy) = G(x)G(y)\,. \tag{3}$$

In the present note we shall give the general solutions of equation (1) in the case where $m \leqslant n$ but where the matrices A, B or the matrix F may be singular. The elements of these matrices are from an arbitrary field K. Let us denote by $\overline{GL}(n)$ the multiplicative semigroup of all square $n\times n$ matrices over K, whereas $GL(n)$ denotes as usual the full group of such non-singular matrices. There exist four possibilities for the multiplicative function $F(A)$:

I F: $GL(n)\to GL(m)$

II F: $GL(n)\to \overline{GL}(m)$

III F: $\overline{GL}(n)\to GL(m)$

IV F: $\overline{GL}(n)\to \overline{GL}(m)$.

We shall deal with the singular cases II-IV. The solution in the case II for $n = m = 2$ has been given by Kucharzewski and Kuczma [2].

We recall that if a function $F(A)$ is a solution of (1) and C is a non-singular matrix then $CF(A)C^{-1}$ is also a solution of (1) and thus it is sufficient to determine the solution $F(A)$ with accuracy to the above similarity relation, i.e. to the choice of a base for F as linear transformation of the vector space K^m.

First we shall state several lemmas which are valid for arbitrary m and n.

Lemma 1. *For any family of commuting idempotent matrices there exist a base in which all these matrices have a diagonal form* (see [1], p. 15).

Let us denote by $\{d_1, \ldots, d_n\}$ a diagonal matrix with the elements $d_1, \ldots, d_n$ on the main diagonal.

Lemma 2. *The following matrices*

$$A_{1\ldots r} = \{\underbrace{1\ldots1}_{r}, 0\ldots0\} \qquad 0 \leqslant r \leqslant n-1 \tag{4}$$

and

$$\begin{cases} R(\varrho) = \{\varrho, 1\ldots1\} \\ S(\varrho) = \left\{\begin{bmatrix}1 & \varrho\\ 0 & 1\end{bmatrix}, 1\ldots1\right\} \\ V_i = \left\{1\ldots1, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, 1\ldots1\right\}, \end{cases} \tag{5}$$

where the submatrix $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ *stands in lines* $i, i+1$, *generate the full semigroup* $\overline{GL}(n)$, *i.e. any matrix* $A \in \overline{GL}(n)$ *is a product of a finite number of such matrices.*

In fact, matrices (5) generate all elementary non-singular matrices and thus the full linear group $GL(n) \subset \overline{GL}(n)$. If a matrix B is singular of a rang r then there exist non-singular matrices P, Q such that

$$B = PA_{1\ldots r}Q\,. \tag{6}$$

P, Q are generated by matrices (5) and thus B is generated by matrices (4) and (5).

Definition 1. A solution $F(A)$ of equation (1) will be called *decomposable* if there exists a base in which it has the form

$$F(A) = \begin{bmatrix}F_1(A) & 0\\ 0 & F_2(A)\end{bmatrix}.$$

i.e. if it is the direct sum of solutions F_1 and F_2.

Definition 2. A solution F of equation (1) will be called *regular* if $F(A) = 0$ for any singular A and $F(A) \in GL(m)$ for any non-singular A.

Obviously any regular solution is a trivial extension of a solution of equation (1) in the case I, by putting $F(A) = 0$ for any singular matrix A.

Lemma 3. *If* $F(A)$ *is a non-decomposable solution then*

$$F(0) = 0 \quad \textit{and} \quad F(E_n) = E_m\,, \tag{7}$$

where E_n, E_m are the unit matrices of order n and m respectively, and 0 is the null matrix.

Proof. The matrices $F(0)$ and $F(E_n)$ are idempotent, since 0 and E_n are idempotent. If $F(0) = J \neq 0$, then, by lemma 1, in a suitable base J has the form

$$J = \{1 \ldots 1, 0 \ldots 0\} \tag{8}$$

(it is obvious that only 0's and 1's may stand on the main diagonal). Any matrix A commutes with the null matrix and thus $F(A)$ commutes with J, what implies that the solution $F(A)$ must be decomposable. The same reasonning leads to this conclusion in the case where $F(E_n) = J \neq 0$.

Remark. In view of (1) any constant solution different from zero is an idempotent matrix and thus it must have the form

$$F(A) = C^{-1}JC, \tag{9}$$

where C is a constant non-singular matrix and J has one of forms (8).

Lemma 4. *The restriction of any non-decomposable solution $F: \overline{GL}(n) \to \overline{GL}(m)$ to the group $GL(n)$ is a solution of the type I and thus if $n \leqslant m$ it has one of forms* (2).

The above statement follows easily from the equation $F(E_n) = E_m$ which implies that $F(AA^{-1}) = E_m$ for any $A \in GL(n)$ and thus $F(A)$ is also a non-singular matrix, and moreover

$$F(A^{-1}) = F^{-1}(A). \tag{10}$$

Relation (10) allows us to state the following

Lemma 5. *If the matrices $A, B \in \overline{GL}(n)$ are similar then the matrices $F(A)$ and $F(B)$ are also similar, provided F is a non-decomposable solution.*

2. In this section we shall determine all solutions of equation (1) in the most general case IV. For the sake of brevity we shall write $\tilde{A}$ instead of $F(A)$. Thus, for $A \in \overline{GL}(n)$ we will have $\tilde{A} \in \overline{GL}(m)$.

By $A_{i_1 \ldots i_r}$ we denote the diagonal matrix with r 1's standing on the diagonal on places $i_1 < \ldots < i_r$. All such matrices form a family of commuting idempotents and, in view of (1), all corresponding $\tilde{A}_{i_1 \ldots i_r}$ form such a family. By Lemma 1, we can choose a base in which they have the diagonal forms

$$\{\varepsilon_1, \ldots, \varepsilon_m\},$$

where, by idempotence, $\varepsilon_i = 0, 1$.

In the sequel we shall assume that the considered solution F is non-decomposable and thus, in particular, that relations (7) hold. We assume also $1 < m \leqslant n$.

Since for the matrices A_i $(i = 1, \ldots, n)$, $A_i A_j = 0$ for $i \neq j$, we have also

$$\tilde{A}_i \tilde{A}_j = 0 \quad \text{for } i \neq j \tag{11}$$