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On the Existence of Solution of Two-point Boundary-value Problem for Second Order Differential Equation

1. In the present note we shall consider a differential equation

$$y''+g(t, y, y') = 0 \tag{1}$$

and the boundary-value problems of the following types:

$$y(0) = a, \quad y'(k) = b, \tag{2}$$

$$y'(k) = b, \quad y(h) = c, \tag{3}$$

$$y(0) = a, \quad y(h) = c, \tag{4}$$

where $a, b, c, k, h (0 < k < h)$ are real numbers.

For the real-valued function $g(t, u, v)$ we assume the Carathéodory conditions (C); i.e., we assume $g(t, u, v)$ to be defined in the set

$$D: \{(t, u, v): 0 \leqslant t \leqslant h, -\infty < u, v < +\infty\},$$

continuous with respect to (u, v) for every fixed $t \in \langle 0, h\rangle$ and Lebesgue integrable with respect to t for every fixed $(u, v) \in R^2$. Moreover we assume that $g(t, u, v)$ satisfies in the set D the following inequality:

$$-G(-u, -v) \leqslant g(t, u, v)-g(t, 0, 0) \leqslant G(u, v) \tag{5}$$

where $G(u, v)$ is the continuous piecewice linear function

$$G(u, v) = \begin{cases} K_1 u+L_1 v & \text{for } u \leqslant 0, v \leqslant 0, \\ K_1 u+L_2 v & \text{for } u \leqslant 0, v \geqslant 0, \\ K_2 u+L_1 v & \text{for } u \geqslant 0, v \leqslant 0, \\ K_2 u+L_2 v & \text{for } u \geqslant 0, v \geqslant 0, \end{cases}$$

(K_1, K_2, L_1, L_2 denote arbitrary real numbers).

By a solution of (1) we mean any absolutely continuous function $y(t)$ defined in $\langle 0, h\rangle$ and satisfying (1) almost everywhere on $\langle 0, h\rangle$.

The purpose of this paper is to prove the following

Theorem 1. *If the function $g(t, u, v)$ satisfies in the set D conditions (C) and inequality (5) then*

1° *the inequality $0 < k < \alpha(L_2, K_2)$ implies the existence of a solution of problem (1), (2),*

2° *the inequality $0 < h-k < \beta(L_1, K_2)$ implies the existence of a solution of problem (1), (3),*

3° *both above inequalities imply the existence of a solution of problem (1), (4), where the constants $\alpha(L, K)$, $\beta(L, K)$ are given by the formulae*

$$\alpha(L, K) = \begin{cases} \dfrac{2}{\sqrt{4K-L^2}} \arccos \dfrac{L}{2\sqrt{K}} & \text{for } 4K-L^2 > 0 \\ \dfrac{2}{\sqrt{L^2-4K}} \operatorname{area\,cosh} \dfrac{L}{2\sqrt{K}} & \text{for } 4K-L^2 < 0,\ L > 0,\ K > 0 \\ \dfrac{2}{L} & \text{for } 4K-L^2 < 0,\ L > 0 \\ \infty & \text{otherwise.} \end{cases}$$

$$\beta(L, K) = \begin{cases} \dfrac{2}{\sqrt{4K-L^2}} \cos \dfrac{L}{2\sqrt{K}} & \text{for } 4K-L^2 > 0 \\ \dfrac{2}{\sqrt{L^2-4K}} \operatorname{area\,cosh} \dfrac{L}{2\sqrt{K}} & \text{for } 4K-L^2 < 0,\ K > 0 \\ \dfrac{-2}{L} & \text{for } 4K-L^2 < 0,\ L < 0 \\ \infty & \text{otherwise.} \end{cases} \tag{6}$$

The above theorem is a generalization of recent theorems concerning the uniqueness and the existence of solutions of analogous problems which have been obtained in a different way by Bailey, Shampine and Waltman (see [1]). (Instead of (5) they assume a more restrictive condition: $-G(\bar{u}-u, \bar{v}-v) \leqslant g(t, u, v) - g(t, \bar{u}, \bar{v}) \leqslant G(u-\bar{u}, v-\bar{v})$).

The proof of Theorem 1 consists of three parts which are given in sections 2, 3, and 4, respectively.

In section 2 we state a theorem concerning contingent equations which is due to A. Lasota. Section 3 contains three lemmas that one can easily obtain from results included in papers [1], [2], [3]. Finally, using results of the preceding sections, in section 4 we prove Theorem 1.

2. Denote by $cf(R^n)$ the metric space of all closed and convex subsets of R^n, where the metric function ϱ is given by the Hausdorff distance of two sets; i.e., for any A, B in $cf(R^n)$,

$$\varrho(A, B) = \max\big(\sup_{x\in B} \delta(x, A), \sup_{x\in A} \delta(x, B)\big),$$

where $\delta(x, A)$ is the Euclidean distance of the point x to the set A.

Let $C^n_{\langle 0,h\rangle}$ denote the space of all continuous mappings of $\langle 0, h\rangle$ into R^n, with the usual norm

$$\|x\|_C = \max|x(t)|.$$

We adopt the following definition (see [5]): a map $H: \langle 0, h\rangle \to cf(R^n)$ is said to be *Lebesgue measurable* if for each closed $A \subset R^n$ the set $\{t: H(t) \cap A \neq \emptyset\}$ is Lebesgue measurable. We assume that the maps

$$F: \langle 0, h\rangle \times R^n \to cf(R^n), \; f: \langle 0, h\rangle \times R^n \to R^n, \; N: C^n_{\langle 0,h\rangle} \to R^n$$

satisfy the following conditions:

(i) F satisfies the Carathéodory conditions, i.e., F is measurable with respect to t for every fixed $x \in R^n$, and F is continuous with respect to x for every fixed $t \in \langle 0, h\rangle$. We assume moreover that $F(t, x)$ is homogeneous in x; i.e., $F(t, \lambda x) = \lambda F(t, x)$ for each real λ, and we suppose that there is a Lebesgue integrable real function $\varphi(t)$, $t \in \langle 0, h\rangle$, such that for every fixed t

$$|y| \leqslant \varphi(t) \text{ for each } y \in \bigcup_{|x|=1} F(t, x).$$

(ii) The map f satisfies the Carathéodory conditions and

$$\lim_{p\to\infty} \frac{1}{p} \int\limits_0^h \sup_{|x|\leqslant p} \delta\big(f(t, x), \quad F(t, x)\big)\, dt = 0.$$

(iii) The map N is continuous and homogeneous; that is $N(\lambda x) = \lambda N(x)$ for each real λ and $x \in C^m_{\langle 0,h\rangle}$.

By a *solution* of the contingent equation

$$x' \in F(t, x) \tag{7}$$

we mean any absolutely continuous function in an interval, say $\langle 0, h\rangle$, into R^n which satisfies (7) almost everywhere.

Now we can state Lasota's theorem already mentioned in Section 1 in the following manner:

Theorem 2. *Assume F, f and N satisfy assumptions (i), (ii) and (iii) respectively. If $x(t) \equiv 0$ is the unique solution of (7) defined on $\langle 0, h\rangle$ which satisfies the homogeneous boundary value condition $N(x) = 0$, then the boundary value problem*

$$x' = f(t, x) \tag{8}$$

$$N(x) = r \tag{9}$$

has at least one solution for each $r \in R^n$.

3. In the proof of Theorem 1 we shall make use of three lemmas which are to be preceeded by the following

Remark. Let $z(t)$ denote a solution of the differential equation

$$z'' + Lz' + Kz = 0 \text{ (L, K are real numbers)}$$

satisfying the condition $z(t_0) \neq 0$, $z'(t_0) = 0$.

Let $\alpha(L, K)$, $\beta(L, K)$ be the respective distances between the point t_0 and the next preceding and next following zero of $z(t)$. If $z(t)$ has not any zero smaller (or larger) than t_0, we set $\alpha(L, K) = \infty$ ($\beta(L, K) = \infty$). The numbers $\alpha(L, K)$ and $\beta(L, K)$ are given (see [1], [2]) by formulae (6).

Furthermore (also from [1], [2]) numbers $\alpha(L_2, K_2), \beta(L_1, K_2)$ are the minimum respective distances between a zero of $z'(t)$ and the next preceding and next following zero of any non trivial solution $z(t)$ of the differential equation

$$z''+G(z, z') = 0 \quad \text{or} \quad z''-G(-z, -z') = 0 .$$

Lemma 1. *If a real number k satisfies the inequality*

$$0 < k < \alpha(L_2, K_2) ,$$

then $y(t) \equiv 0$ is the only function defined on $\langle 0, h\rangle$, with absolutely continuous derivative $y'(t)$ satisfying the conditions

(10) $$-G(y, y') \leqslant y'' \leqslant G(-y, -y') ,$$

(11) $$y(0) = y'(0) = 0 .$$

Proof. Let us first assume the function $y(t)$ satisfies assumptions of our lemma and also condition $y'(0) = 0$. Thus $y(t)$ is a solution of the differential inequality

$$|y''| \leqslant A|y|+B|y'|$$

$$(A = \max(|K_1|, |K_2|),\ B = \max(|L_1|, |L_2|))$$

satisfying the condition $y(0) = y'(0) = 0$. Hence we have $y(t) \equiv 0$ for $t \in \langle 0, h\rangle$.

Thus in order to prove Lemma 1 it is sufficient to show that the case $y'(0) \equiv 0$ is impossible. Suppose the contrary $y'(0) > 0$ (the case $y'(0) < 0$ is analogous).

From assumption (10) it follows that $y(t)$ is a non-trivial solution of the differential inequality

$$y''+G(y, y') \geqslant 0 ,$$

satisfying condition (11) and without loss of generality we can assume that k is the smallest positive zero of $y'(t)$. Denoting by $z(t)$ a solution of the differential equation

$$z''+G(z, z') = 0$$

satisfying conditions $z(0) = 0$, $z'(0) = y'(0) > 0$, we have (see [3], Lemma 2) the inequality

$$\operatorname{arc\,tg}\frac{y}{y'} \leqslant \operatorname{arc\,tg}\frac{z}{z'} .$$

But condition (11) implies that

$$\operatorname{arc\,tg}\frac{z(k)}{z'(k)} = \frac{\pi}{2} .$$

Hence we have $z'(k) = 0$ which is a contradiction (see the above Remark) with the assumption that $k < (L_2, K_2)$. Thus we have $y'(0) = 0$ and this completes the proof.

In a similar way we can prove the following

Lemma 2. *If real numbers k, h satisfy the inequality*

$$0 < h - k < \beta(L_1, K_2),$$

then $y(t) \equiv 0$ is the only function defined on $\langle 0, h\rangle$ with absolutely continuous derivative $y'(t)$ satisfying inequality (10) *in the interval $\langle 0, h\rangle$ and condition*

$$y'(k) = y(h) = 0. \tag{12}$$

Lemma 1 and Lemma 2 enable us to obtain the following

Lemma 3. *If a real number h satisfies the inequality*

$$0 < h < \alpha(L_2, K_2) + \beta(L_1, K_2),$$

then $y(t) \equiv 0$ is the only function defined on $\langle 0, h\rangle$ with absolutely continuous derivative $y'(t)$ satisfying inequality (10) *in the interval $\langle 0, h\rangle$ and condition*

$$y(0) = y(h) = 0. \tag{13}$$

4. Proof of Theorem 1. Setting $x_1 = y$, $x_2 = x_1'$ we can replace differential equation (1) by a system of two first order differential equations

$$x_1' = x_2, \; x_2' = -g(t, x_1, x_2).$$

Now equation (1) and conditions (2), (3), (4) can be written shortly in the vector notation as

$$x' = f(t, x), \tag{13}$$

$$N_i(x) = r_i \; (i = 1, 2, 3), \tag{14}$$

where $x = (x_1, x_2) \in R^2$ and

$$\begin{aligned}
&f\colon \langle 0, h\rangle \times R^2 \ni (t, x_1, x_2) \to (x_2, -g(t, x_1, x_2)) \in R^2\\
&N_1\colon C^2_{\langle 0,h\rangle} \ni (x_1(t), x_2(t)) \to (x_1(0), x_2(k)) \in R^2\\
&N_2\colon C^2_{\langle 0,h\rangle} \ni (x_1(t), x_2(t)) \to (x_1(h), x_2(k)) \in R^2\\
&N_3\colon C^2_{\langle 0,h\rangle} \ni (x_1(t), x_2(t)) \to (x_1(0), x_1(h)) \in R^2\\
&r_1 = (a, b), \; r_2 = (c, b), \; r_3 = (a, c).
\end{aligned}$$

For $t \in \langle 0, h\rangle$ and $(x_1, x_2) \in R^2$ we adopt the definition

$$F(t, x_1, x_2) = \{(y_1, y_2) \in R^2 \colon y_1 = x_2, \; -G(x_1, x_2) \leqslant y_2 \leqslant G(-x_1, -x_2)\}.$$

Since $F(t, x_1, x_2)$ is a convex and closed subset of R^2, we have the mapping

$$F\colon \langle 0, h\rangle \times R^2 \ni (t, x_1, x_2) \to F(t, x_1, x_2) \in cf(R^2).$$

It is easily seen that the map F satisfies condition (i).