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On the Existence of Solutions of the Floquet Problem for Ordinary Differential Equations

Consider the system of differential equations

(1) $$x' = f(t, x)$$

with a boundary condition

(2) $$x(0) + \lambda x(T) = r \qquad (\lambda > 0),$$

where x is a vector of R^n and the n-vector function $f(t, x)$ satisfies on $[0, T] \times R^n$ the Carathéodory conditions:

(C) $f(t, x)$ is continuous in x for each $t \in [0, T]$,
$f(t, x)$ is Lebesgue measurable in t for each $x \in R^n$.

By a solution of (1) we mean any absolutely continuous function $x(t)$ satisfying (1) almost everywhere on $[0, T]$. For $x \in R^n$, $|x|$ denotes the Euclidean norm of x.

A. Lasota and Z. Opial [2, Theorem 3] have proved that if $f(t, x)$ satisfies (C) and

(3) $$|f(t, x) - f(t, y)| \leqslant p(t)|x - y|, \quad \text{for } x, y \in R^n, \ t \in [0, T],$$

where

$$\int_0^T p(s)\,ds < \pi,$$

then the boundary-value problem (1), (2) has exactly one solution for each $r \in R$.

This note presents some generalization of the mentioned result. In Section 1 we give the statement of results (Theorems 1, 2). Section 2 contains a theorem concerning a certain inequality (Theorem 3) which is used in proofs of Theorems 1 and 2 that are given in Section 3. In the last Section we give an example which shows that the estimate obtained in Theorem 3 is the best possible.

1. Theorem 1. *Let* $f(t, x)$ *satisfy* (3) *where* $p(t)$ *is a positive summable function such that*

$$\int_0^T p(s)\,ds < \sqrt{\pi^2+\ln^2\lambda}\,. \tag{4}$$

Then a solution of (1), (2), *if exists, is unique for any* $r \in R^n$ *and any* $\lambda > 0$.

Theorem 2. *Assume that* $f(t, x)$ *satisfies* (C) *and*

$$|f(t, x)| \leqslant p(t)|x| + q(t) \quad \textit{for } (t, x) \in [0, T] \times R^n\,, \tag{5}$$

where functions $p(t)$ *and* $q(t)$ *are summable on* $[0, T]$.

If $p(t)$ *is nonnegative and satisfies* (4), *then the problem* (1), (2) *has at least one solution.*

From Theorems 1 and 2 it follows immediately the following

Corrolary (cf. [2], Theorem 3). *If* $f(t, x)$ *satisfies* (C) *and* (3), $p(t)$ *is nonnegative, summable on* $[0, T]$ *and satisfies* (4), *then the problem* (1), (2) *has exactly one solution.*

2. Consider the inequality

$$|x'| \leqslant p(t)|x| \tag{6}$$

with the homogeneous boundary condition

$$x(0) + \lambda x(T) = 0 \qquad (\lambda > 0)\,. \tag{7}$$

Theorem 3. *If* $p(t)$ *is a nonnegative function, summable on* $[0, T]$ *and satisfying* (4), *then* $x(t) \equiv 0$ *is the only absolutely continuous function satisfying conditions* (6), (7).

Proof. From (6) and the Gronwall inequality it follows that

$$|x(t)| \leqslant |x(t_0)| \cdot \exp\left(\int_{t_0}^{t} p(s)\,ds\right).$$

Hence if $x(t_0) = 0$, then $x(t) \equiv 0$. Thus if an absolutely continuous function $x(t)$ satisfies (6), then $x(t) \equiv 0$ or $x(t) \neq 0$ on $[0, T]$.

Let $x(t)$ be an absolutely continuous function satisfying (6) and (7). Assume that $x(t) \neq 0$ on $[0, T]$. Let $\varepsilon > 0$ be so small that $p^*(t) = p(t) + \varepsilon$ satisfies (4). By a change of the independent variable

$$\tau = \int_0^t p^*(s)\,ds$$

we reduce (6), (7) to the form

(8) $$|y'(\tau)| \leqslant |y(\tau)|\,,$$

(9) $$y(0)+\lambda y(\tau^*)=0 \quad \Big(\tau^* = \int_0^T p^*(s)\,ds\Big).$$

Let $\varrho(\tau) = |y(\tau)|$, $e(\tau) = \big(1/\varrho(\tau)\big)y(\tau)$. Then (8) and (9) give

(10) $$\big(\varrho'(\tau)\big)^2 + \big(\varrho(\tau)\big)^2|e'(\tau)|^2 \leqslant \big(\varrho(\tau)\big)^2\,,$$

(11) $$\varrho(0) = \lambda\varrho(\tau^*), \qquad e(0) = -e(\tau^*)\,.$$

Since

$$\int_0^{\tau^*} |e'(s)|\,ds$$

is the length of an arc joining the points $e(0)$ and $e(\tau^*)$, from (11) it follows that

$$\int_0^{\tau^*} |e'(s)|\,ds \geqslant \pi\,.$$

By (10), $|e'(\tau)| \leqslant \sqrt{1-\big(\varrho'(\tau)/\varrho(\tau)\big)^2}$. Integrating this inequality we obtain

$$\int_0^{\tau^*} |e'(s)|\,ds \leqslant \int_0^{\tau^*} \sqrt{1-(\varrho'/\varrho)^2}\,ds\,.$$

Hence

(12) $$\pi \leqslant \int_0^{\tau^*} \sqrt{1-\big(\varrho'(s)/\varrho(s)\big)^2}\,ds\,.$$

Observe that

(13) $$\int_0^{\tau} \sqrt{1-\big(\varrho'(s)/\varrho(s)\big)^2}\;ds \leqslant \sqrt{\tau^2-\ln^2\big(\varrho(\tau)/a\big)} \quad \big(a = \varrho(0)\big)\,.$$

In fact, from the obvious inequality

$$\big((1/\tau)\ln(\varrho/a) - (\varrho'/\varrho)\big)^2 \geqslant 0$$

it follows that

$$\big(1-(\varrho'/\varrho)^2\big)\big(1-(1/\tau^2)\ln^2(\varrho/a)\big) \leqslant \big(1-(\varrho'/\tau\varrho)\ln(\varrho/a)\big)^2\,,$$

hence

$$\sqrt{1-(\varrho'/\varrho)^2} \leqslant \big(\tau-(\varrho'/\varrho)\ln(\varrho/a)\big)/\sqrt{\tau^2-\ln^2(\varrho/a)}\,.$$

A quadrature over $[0,\tau]$ gives (13).

By (12) and (13), $\pi \leqslant \sqrt{\tau^{*2}-\ln^2\big(\varrho(\tau^*)/a\big)}$ and therefore,

$$\sqrt{\pi^2+\ln^2\lambda} \leqslant \tau^* = \int_0^T p^*(s)\,ds\,,$$

which gives a contradiction because $p^*(t)$ satisfies (4). Hence $x(t) \equiv 0$, which completes the proof of Theorem 3.

3. Proof of Theorem 1. Let $\bar{x}(t)$, $\bar{\bar{x}}(t)$ be solutions of (1) satisfying (2). By (3), $x(t) = \bar{x}(t) - \bar{\bar{x}}(t)$ satisfies (7). So from Theorem 3 it follows that $x(t) \equiv 0$. Hence the solution of the problem (1), (2) is necessarily unique.

Proof of Theorem 2. The proof consists in showing that this theorem is a special case of a general result proved by A. Lasota [1].

Denote by $cf(R^n)$ the metric space of all non-empty, closed and convex subsets of R^n. The (Hausdorff) metric d in $cf(R^n)$ is defined by

$$d(A, B) = \max\big(\sup_{x \in B} r(x, A),\ \sup_{x \in A} r(x, B)\big) \qquad A, B \in cf(R^n),$$

where $r(x, A)$ denotes the Euclidean distance of x from A.

Define the mapping $F(t, x)$ of $[0, T] \times R^n$ into $cf(R^n)$ by

$$F(t, x) = \{u : u \in R^n,\ |u| \leqslant p(t)|x|\}.$$

$F(t, x)$ is continuous with respect to x for each $t \in [0, T]$ and measurable in the sense of A. Pliś [3] in t for each fixed x (i.e. for each closed $A \subset R^n$, the set $\{t : A \cap F(t, x) \neq \varnothing\}$ is Lebesgue measurable). Moreover $F(t, ax) = aF(t, x)$ for all real a.

From Theorem 3 it follows that the problem

$$x'(t) \in F\big(t, x(t)\big),\quad x(0) + \lambda x(T) = 0 \quad (\lambda > 0)$$

has only the trivial solution. By (5), $d(f(t, x), F(t, x)) \leqslant q(t)$. Hence, from Theorem 2.1 of [1] we conclude that the assertion of Theorem 2 holds.

4. The estimate (4) for $p(t)$ is optimal in the following sense: if (4) is replaced by the weaker condition

$$\int_0^T p(s)\,ds = \sqrt{\pi^2 + \ln^2\lambda}, \tag{14}$$

then Theorems 1, 2, 3 fail to be true.

In fact, consider the system of two differential equations

$$x_1' = -\Big(\frac{1}{\pi}\ln\lambda\Big)x_1 + x_2, \qquad x_2' = -x_1 - \Big(\frac{1}{\pi}\ln\lambda\Big)x_2. \tag{15}$$

The right-hand side of (15) satisfies (3) and (5) with $q(t) = 0$, $p(t) = \frac{1}{\pi}\sqrt{\ln^2\lambda + \pi^2}$. It is easy to verify that the general solution of (15) is of the form

$$x_1(t) = \exp\Big(-\frac{t}{\pi}\ln\lambda\Big)(C_1\cos t + C_2\sin t),$$

$$x_2(t) = \exp\Big(-\frac{t}{\pi}\ln\lambda\Big)(-C_1\sin t + C_2\cos t),$$

where C_1, C_2 are arbitrary constants. The vector $x(t) = (x_1(t),\ x_2(t))$ satisfies

$$|x'(t)| = \frac{1}{\pi}\sqrt{\ln^2\lambda + \pi^2}|x(t)|\,.$$

Obviously $p(t)$ satisfies (14) with $T = \pi$. Observe that

$$x(0) + \lambda x(\pi) = 0$$

holds for all values of C_1, C_2 but for $|r| \neq 0$ the problem

$$x(0) + \lambda x(\pi) = r$$

has no solution.

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