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Convex Sets and the Modulus of Continuity

1. By a well-known theorem of Lebesgue [4] a continuous function f defined on a closed set X in R^m possesses a continuous extension to the whole space, bounded by the same bounds. This has been generalized by Tietze [7] to metric spaces and by Urysohn [8] to the normal ones. An explicit formula for a space with distance ϱ has been given by Hausdorff [2],

$$(1) \quad f_H(x) = \inf_{x' \in E} \left[f(x') + \frac{\varrho(x, x')}{\varrho(x, E)} - 1 \right].$$

In the approximation theory an important role is played by the modulus of continuity

$$(2) \quad \omega(\delta) = \omega(\delta, f) = \sup \{ |f(x'') - f(x')| : |x'' - x'| \leq \delta; x', x'' \in E \}.$$

ω is clearly increasing with δ and $\omega(0+) = 0$ if and only if f is uniformly continuous. It is natural to raise the problem of extending f with the same modulus of continuity *). This is impossible in general, so instead of ω some substitutes are being used (e.g., [6]). J. Siciak and W. Kleiner have stated the following conjecture:

Let E be a closed subset of a normed linear space Ω . Then every function defined and uniformly continuous on E can be extended to a function on Ω with the same modulus of continuity if only if E is convex.

I will prove this conjecture for Hilbert spaces, where a simple continuation formula may also be given.

*) For special ω , e.g. that of Lipschitz or Hölder type, strong results have been obtained by McShane [5], Kirszbraun [3] and Valentine [9].

