ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
Nr CCIII PRACE MATEMATYCZNE, ZESZYT 13 1969

Andrzej Pelczar

On Some Iterative-differential Equations II

The present paper is the second part of the paper [1], in which were proved theorems concerning the existence and the uniqueness of the solutions of the equation

$$(\ast)\qquad y(x) = c + \int_0^x f\big(t, y(t), y(y(t))\big)\,dt$$

(cf. (3) in [1]), equivalent to the equation

$$\frac{dy}{dx} = f\big(t, y(t), y(y(t))\big)$$

with the initial condition

$$y(0) = c$$

(cf. (1)-(2) in [1]).

In this paper we give some other theorems concerning $(\ast)$.

We continue the same numeration of sections, theorems and formulas which had been introduced in the first part [1].

6. Theorem 3. *Let us suppose that*

1. $|f(x, y, u)| \leqslant K$,

2. *for each* c *belonging to some closed interval* U, *there exists at most one solution* $y = y(x)$ *of equation* $(\ast)$,

3. *a sequence* $\{c_n\}$, $c_n \in U$, *converges to* c *as* $n \to \infty$,

4. *for each* n *there exists the (unique) solution* $y_n = y_n(x)$, $0 \leqslant y \leqslant a$ *of the equation*

$$(21)\qquad y(x) = c_n + \int_0^x f\big(t, y(t), y(y(t))\big)\,dt\,.$$

Then the sequence $\{y_n\}$ *is uniformly convergent in* $\langle 0, a\rangle$ *and the limit* $y(x) = \lim y_n(x)$ *is the (unique) solution of equation* $(\divideontimes)$.

Proof. The functions y_n are equicontinuous; hence there exists a subsequence $\{y_{a_n}\}$ uniformly convergent to some function $y = y(x)$. It is easy to see that $y(x)$ is a solution of $(\divideontimes)$. The solution of $(\divideontimes)$ must be unique and therefore each subsequence of $\{y_n\}$ is uniformly convergent to this solution $y(x)$. Then $\{y_n\}$ uniformly converges to $y = y(x)$.

Corollary. *If the assumptions of the uniqueness of solutions of* $(\divideontimes)$ *is satisfied, then solutions depend continuously on initial conditions.*

7. **Remark 1.** In [1] we assumed that $c \geqslant Ka$ (cf. (10)). In the case $0 \leqslant f \leqslant K$, we can assume only $c \geqslant 0$.

Theorem 4. *If* $0 \leqslant f \leqslant K$, $c+Ka \leqslant a, c \geqslant 0$ *and* $f = f(x, y, u)$ *is increasing with respect to* y, u, *then there exist the maximal and the minimal solutions of the problem* (1)-(2) (*in other words*: *of equation* $(\divideontimes)$) *in the class of increasing functions* $z \in C^1(\langle 0, a\rangle)$, $0 \leqslant z \leqslant a$.

Theorem 5. *Let us suppose the assumptions of Theorem 4. If an increasing function* $v = v(x)$ *belonging to* $C^1(\langle 0, a\rangle)$, $v(0) \leqslant c$, $0 \leqslant v \leqslant a$ *satisfies the inequality*

$$\frac{dv}{dx} \leqslant f\big(x, v(x), v(v(x))\big), \tag{22}$$

then $v(x) \leqslant \hat{y}(x)$ *for* $x \in \langle 0, a\rangle$, *where* $\hat{y} = \hat{y}(x)$ *is the maximal solution of* $(\divideontimes)$.

The proofs of Theorems 4 and 5 follow directly from the results of [2]. Indeed, let us denote

$$Q = \Big\{y \in C^1(\langle 0, a\rangle) : y \text{ is increasing}, \ |y'| \leqslant K,$$
$$0 \leqslant y(x) \leqslant c + \int_0^x f\big(t, y(t), y(y(t))\big)\,dt\Big\}$$

It is easy to see that Q is non-empty since $y = 0$ belongs to Q.

Let us put

$$F : C^1(\langle 0, a\rangle) \ni y \to F(y) \in C^1(\langle 0, a\rangle)$$

where

$$F(y)(x) \overset{\mathrm{df}}{=} c + \int_0^x f\big(t, y(t), y(y(t))\big)\,dt.$$

If we introduce the following relation: $y \leqslant z$ if and only if $y(x) \leqslant z(x)$ for each $x \in \langle 0, a\rangle$, then $C^1(\langle 0, a\rangle)$ will be a partially ordered set. It is easy to prove that for $Q \subset C^1(\langle 0, a\rangle)$ and the mapping F all assumptions of Theorem 1 from [2] are satisfied. Hence there exists the maximal solution (in the class of increasing functions) of the equation $y = F(y)$. On the same way, using Theorem 2 from [2], we can prove that there exists the minimal solution (increasing) of the equation $y = F(y)$.

Hence there exist the maximal and the minimal solutions of $(\ast)$ in the class of increasing functions belonging to $C^1(\langle 0, a\rangle)$, and the proof of Theorem 4 is complete.

In virtue of Remark 1 in [2], the above reasoning proves Theorem 5 too.

Remark 2. Let us suppose the assumptions of Theorem 4 and let w be an arbitrary function of the class $C^1(\langle 0, a\rangle)$, $w(0) \leqslant c$, $0 \leqslant w \leqslant a$, satisfying inequality (22). It is easy to see that

$$v(x) = c + \int_0^x f\big(t, w(t), w(w(t))\big)\,dt$$

fulfils all assumptions of Theorem 5. Hence $v \leqslant \hat{y}$ and in the consequence $w \leqslant \hat{y}$.

Corollary. *The maximal solution* $\hat{y} = \hat{y}(x)$ *of* $(\ast)$ *in the class of increasing functions belonging to* $C^1(\langle 0, a\rangle)$ *is the maximal solution in the class* $C^1(\langle 0, a\rangle)$.

Remark 3. On an analogous way we can prove that the minimal solution of $(\ast)$ in the class of increasing functions belonging to $C^1(\langle 0, a\rangle)$ is the minimal solution in the class $C^1(\langle 0, a\rangle)$.

Therefore we proved the following

Theorem 6. *If* $0 \leqslant f \leqslant K$, $c+Ka \leqslant a$, $c \geqslant 0$ *and* $f = f(x, y, u)$ *is increasing with respect to* y, u, *then there exist the maximal and the minimal solutions of the problem* (1)-(2) (*resp., of equation* $(\ast)$) *in the class of functions* $z \in C^1(\langle 0, a\rangle)$, $0 \leqslant z \leqslant a$.

REFERENCES

1] A. **Pelczar**, *On some iterative-differential equations I*, Zeszyty Naukowe UJ, Prace Matematyczne 12 (1968), 53—56.

[2] A. **Pelczar**, *On invariant points of monotone transformations in partially ordered spaces*, Ann. Polon. Math. 7 (1965), 49—53.