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# Analyticity and Separate Analyticity of Functions Defined on Lower Dimensional Subsets of $C^n$

## Introduction

The famous Hartogs' theorem ([3], [8]) says that if $f(z) = f(z_1, \ldots, z_n)$ is a function defined in a domain $D$ in the space $C^n$ of $n$ complex variables $z_k = x_k + iy_k (k = 1, \ldots, n)$ and $f$ is analytic in each variable $z_k$ separately when the other variables are given arbitrary fixed values, then $f$ is analytic in $D$.

If $u(x) = u(x_1, \ldots, x_n)$ is a function defined in a domain $D$ in the space $R^n$ of $n$ real variables $x_k$ $(k = 1, \ldots, n)$ and $u$ is analytic in each variable $x_k$ separately, then $u$ is not, in general, an analytic function in $D$, even if we in addition assume that $u$ is of class $(C^\infty)$ in $D$. A corresponding example is given by

$$u(x_1, x_2) = x_1 x_2 \exp\left(-\frac{1}{x_1^2 + x_2^2}\right), \quad u(0, 0) = 0, \quad (x_1, x_2) \in R^2.$$

It is however possible to generalize Hartogs' theorem for the following class of functions of $n$ real variables. Let $E$ be a subset of $R^n$. We identify $R^n$ with the subset $\{z \in C^n : y_k = 0, k = 1, \ldots, n\}$ of $C^n$. Then $E$ may be considered as a subset of $C^n$. Let $D$ be a domain in $R^n$. Let $L_D$ denote the class of all the functions $f$ defined in $D$ such that for every $x^0 = (x_1^0, \ldots, x_n^0) \in D$ there exists a polydisc $P(x^0, r) = \{z \in C^n : |z_k - x_k^0| < r_k, k = 1, \ldots, n\}$ such that for fixed real $\xi_1, \ldots, \xi_n$ where $x_k^0 - r_k < \xi_k < x_k^0 + r_k$, the function $f(\xi_1, \ldots, \xi_{j-1}, x_j, \xi_{j+1}, \ldots, \xi_n)$ $x_j^0 - r_j < x_j < x_j^0 + r_j$, is continuable to an analytic function in the disc $|z_j - x_j^0| < r_j$ $(j = 1, \ldots, n)$. The polydisc $P$ may depend on $x^0$ and on $f$. Every function $f \in L_D$ is of course analytic in each variable $x_j$ separately. The functions of the class $L_D$ appeared useful in the study of the following topics: analyticity of distribution kernels, the edge of the wedge theorem, bounded representation of the classical Lie groups on Hilbert space, Feynman integrals, analyticity of solutions for partial differential equations (see Introduction to [5]).

We have proved (in § 3) a theorem which implies the following.

(I) *If $D$ is a domain in $R^n$ then every function $f \in L_D$ is analytic in $D$.*

Corollary. *If $h(x, u) = h(x_1, \ldots, x_p, u_1, \ldots, u_q)$ is a function defined in a domain $D \subset R^{p+q}$ and harmonic with respect to $x$ and $u$ separately then $h$ is harmonic in $D$.*

Theorem (I) generalizes results concerning separate analyticity of real functions due to P. Lelong [13] and F. E. Browder [4] ,where the analyticity has been proved only for those functions $f \in L_D$ which are assumed to satisfy some boundedness conditions. The result formulated in Corollary has been first proved in [13] (see also [1] and [2]).

The method of proof of (I) is based on Leja's polynomial lemma [10] and on the expansion of functions analytic on a line interval into the series of Chebyshev polynomials (see § 2). Leja ([11], [12]) used his lemma to give a new proof and a generalization of the Hartogs' main lemma (for functions of one variable) which implies easily Hartogs' theorem (for functions of two complex variables). We used the Leja's reasoning to get Theorem 1 (§ 1) which generalizes Hartogs' main lemma (for functions of $n$ variables, $n \geqslant 1$). Our Theorem 1 is very akin to Theorem 10 in [13]. In fact in our proof of (I) we might use Theorem 10 of Lelong [13] instead of Theorem 1. We preferred to use Theorem 1 because its proof is much simpler than that of Theorem 10 in [13].

In § 4 we give generalizations of recent results due to Cameron and Storvick [6] concerning analytic continuation of functions defined on lower dimensional subsets of $C^n$. Two examples of our results in this direction are given by

(II) *Let $G_k = \{w_k : |w_k + \sqrt{w_k^2 - 1}| < R_k\}$ $(R_k > 1)$ be an ellipse in the complex $w_k$-plane with foci $-1$ and $+1$ and with the halfaxes $\frac{1}{2}(R_k + 1/R_k)$ and $\frac{1}{2}(R_k - 1/R_k)$. Let $f$ be defined in*

$$X = (G_1 \times F_2 \times \ldots \times F_n) \cup \ldots \cup (F_1 \times \ldots \times F_{n-1} \times G_n),$$

*where $F_k$ denotes the interval $[-1, 1]$ in the $w_k$-plane. Let $f$ be analytic with respect to $w_k \in G_k$ for every fixed $(u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_n) \in F_1 \times \ldots \times F_{k-1} \times F_{k+1} \times \times \ldots \times F_n$. Then $f$ is continuable to an analytic function $\tilde{f}$ in*

$$\Omega = \Big\{w \in C^n : \prod_{k=1}^{n} |w_k + \sqrt{w_k^2 - 1}|^{\beta_k} < 1\Big\}, \qquad \beta_k = 1/\log R_k .$$

*$\Omega$ is the envelope of holomorphy of $X$ and $\sup_X |f| = \sup_\Omega |\tilde{f}|$.*

(III) *Let $E$ be a compact subset with positive logarithmic capacity of a domain $D$ in $z$-plane. Let $f(z, w)$ be defined in*

$$X = (D \times [-1, 1]) \cup (E \times C)$$

*such that f is analytic with respect to $w \in C$ for each fixed $z \in E$ and $f$ is analytic with respect to $z \in D$ for each fixed $u \in [-1, 1]$. Then $f$ is continuable to an analytic function in $D \times C$.*

In [6] the set $X$ was more restrictive and the local boundedness of $f$ was assumed.

## 1. A generalization of the fundamental lemma of Hartogs

The following lemma will play a basic rọle in our further study.

Polynomial lemma. *Let $E = E_1 \times \ldots \times E_n$, where $E_k$ is a continuum in the complex $z_k$-plane. Let $M(z)$ be a real function defined in $E$ such that*

$$0 < M(z) < +\infty, \quad z \in E. [1]$$

*Let $\mathcal{F}$ be a family of the polynomials $P(z)$ in $n$ complex variables $z = (z_1, \ldots, z_n)$ such that*

$$|P(z)| \leqslant M(z), \quad z \in E, \quad P \in \mathcal{F}.$$

*Then for every $\varepsilon > 0$ and for every $z^0 \in E$ there are two positive numbers $K = K(z^0, \varepsilon)$ and $\delta = \delta(z^0, \varepsilon)$ such that*

$$(1) \qquad |P(z)| \leqslant K_\varepsilon e^{\varepsilon \deg P}, \; \|z - z^0\| \leqslant \delta, \; P \in \mathcal{F},$$

*where $\deg P$ is the largest sum of exponents occurring in a monomial term of $P$.*

If $n = 1$ this lemma is due to F. Leja [10]. If $n \geqslant 2$ it may easily be proved by the induction (see [14]).

We say that a subset $E \subset C^n$ satisfy the *condition* $(L)$ if for every point $z^0 = (z_1^0, \ldots, z_n^0) \in E$ there exists a continuum $E_k$ in the complex plane $(z_k)$ such that $z_k^0 \in E_k$ and $E_1 \times \ldots \times E_n \subset E$.

Theorem 1. *Let $\Omega$ be an open set in $C^m$ and let $E$ be a compact subset of $\Omega$ satisfying the condition $(L)$. Let $f_\nu(z) = f_{\nu_1 \ldots \nu_m}(z_1, \ldots, z_n)$ $(\nu_k = 0, 1, \ldots;$ $k = 1, \ldots, m)$ be a multiple sequence of holomorphic functions in $\Omega$ such that*

(i) *$\{f_\nu\}$ is uniformly bounded on every compact subset of $\Omega$,*

(ii) $\limsup\limits_{|\nu| \to \infty} \left\| \sqrt[|\nu|]{f_\nu(z) R^\nu} \right| \leqslant 1, \; z \in E,$

*where $|\nu| = \nu_1 + \ldots + \nu_m$ and $R^\nu = R_1^{\nu_1} \ldots R_m^{\nu_m}$, $R_k > 0$ $(k = 1, \ldots, m)$.*

*Then for every $\varepsilon > 0$ there exist a positive number $M = M(\varepsilon)$ and an open subset $\Omega_\varepsilon$ of $\Omega$ such that $E \subset \Omega_\varepsilon$ and*

$$|f_\nu(z) R^\nu| \leqslant M e^{|\nu|\varepsilon}, \quad z \in \Omega_\varepsilon, \quad |\nu| \geqslant 0.$$

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[1] It is sufficient to assume that $M(z) < +\infty$ holds "almost everywhere on $E$ with respect to every $z^0 \in E$" (for details see [10]; see also [7]).

Proof. Let $0 < r < \operatorname{dist}(E, \partial\Omega)$. Let $z^0$ be an arbitrary fixed point of $E$. Let

(2) $$f_\nu(z)R^\nu = \sum_{k=0}^{\infty} Q_{\nu k}(z), \quad \|z-z^0\| \leqslant r,$$

be the expansion of $f_\nu(z)R^\nu$ into the series of homogeneous polynomials with respect to $z_1 - z_1^0, \ldots, z_n - z_n^0$ of respective degrees $k$ (so $Q_{\nu k}(z^0 + at) = t^k Q_{\nu k}(z^0 + a)$ for $t \in C$ and $a = (a_1, \ldots, a_n) \in C^n$; if $n = 1$ then $\sum_{k=0}^{\infty} Q_{\nu k}(z) = \sum_{k=0}^{\infty} q_{\nu k}(z_1 - z_1^0)^k$ is the Taylor series of $f_\nu(z)R^\nu$).

Put $M_1 = \sup_{|\nu|\geqslant 0}(\sup_{\|z-z^0\|\leqslant r} |f_\nu(z)|)$. Then by (i) $M_1 < +\infty$. By the Cauchy inequalities

$$|Q_{\nu k}(z^0 + a)| \leqslant \frac{M_1 R^\nu}{r^k}, \quad a \in C^n, \quad \|a\| = 1, k \geqslant 0, |\nu| \geqslant 0,$$

whence

(3) $$\left|\sum_{k=|\nu|+1}^{\infty} Q_{\nu k}(z)\right| \leqslant M_1 R^\nu \frac{(\varrho/r)^{|\nu|+1}}{1-(\varrho/r)}, \quad \|z-z^0\| \leqslant \varrho, \quad 0 < \varrho < r.$$

By (ii) for $\varepsilon > 0$ and $z \in E$ there exists a positive number $M(z, \varepsilon)$ such that

(4) $$|f_\nu(z)R^\nu| \leqslant M(z, \varepsilon)e^{|\nu|\varepsilon}, \quad |\nu| \geqslant 0.$$

$P_\nu(z) = \sum_{k=0}^{|\nu|} Q_{\nu k}(z)$ is a polynomial in $z_1, \ldots, z_n$ of degree at most $|\nu|$. By (2), (3) and (4)

$$|P_\nu(z)| \leqslant M(z, \varepsilon)e^{|\nu|\varepsilon} + M_1 \frac{\varrho}{r-\varrho}(R_1\varrho/r)^{\nu_1}\ldots(R_m\varrho/r)^{\nu_m}, \quad z \in E, \quad \|z-z^0\| \leqslant \varrho < r,$$

whence

$$|e^{-|\nu|\varepsilon}P_\nu(z)| \leqslant M(z, \varepsilon) + M_1\varrho/(r-\varrho), \quad z \in E, \quad \|z-z^0\| \leqslant \varrho, \quad |\nu| \geqslant 0,$$

where $0 < \varrho < \min(r, r/R_1, \ldots, r/R_m)$. Therefore by the polynomial lemma there are positive numbers $K = K(z^0, \varepsilon)$ and $\delta = \delta(z^0, \varepsilon)$ such that

$$|P_\nu(z)| \leqslant Ke^{2|\nu|\varepsilon}, \quad \|z-z^0\| \leqslant \delta, \quad |\nu| \geqslant 0.$$

Hence in virtue of (2) and (3)

$$|f_\nu(z)R^\nu| \leqslant Ke^{2|\nu|\varepsilon} + M_1 R^\nu \frac{(\varrho_1/r)^{|\nu|+1}}{1-(\varrho_1/r)}, \quad \|z-z^0\| \leqslant \varrho_1, \quad |\nu| \geqslant 0,$$

where $0 < \varrho_1 = \varrho_1(z^0, \varepsilon) < \min(r, r/R_1, \ldots, r/R_m, \delta(z^0, \varepsilon))$. Finally

$$|f_\nu(z)R^\nu| \leqslant \big(K + M_1\varrho_1/(r-\varrho_1)\big)e^{2|\nu|\varepsilon}, \quad \|z-z^0\| \leqslant \varrho_1, \quad |\nu| \geqslant 0.$$

Application of the Heine-Borel theorem ends the proof.

Corollary 1.1. (Fundamental lemma of Hartogs). *Let $\{f_\nu\}$ be a multiple sequence of analytic functions uniformly bounded on every compact subset of an open set $\Omega \subset C^n$. Let*

$$\limsup_{|\nu|\to\infty} \sqrt[|\nu|]{|f_\nu(z)R^\nu|} \leqslant 1, \quad z \in \Omega .$$

*Then for every compact subset $G$ of $\Omega$ and for every $\varepsilon > 0$ there exists a positive number $M = M(G, \varepsilon)$ such that*

$$|f_\nu(z)R^\nu| \leqslant Me^{|\nu|\varepsilon}, \quad z \in G, \quad |\nu| \geqslant 0 .$$

The lemma given in this Corollary was formulated by Hartogs in a slightly different (but equivalent) form. He used the lemma to prove that a function of $n$ complex variables analytic in an open set $\Omega \subset C^n$ in each variable separately is analytic in $\Omega$. His method of proof was based essentially on the theory of subharmonic functions.

A generalization of Hartogs' lemma (for $n = 1$) of the type given by Theorem 1 was first offered by F. Leja in [11] (see also [12]). The reasoning used by us to prove Theorem 1 is a simple modification of the reasoning used by Leja in [11] or in [12].

Theorem 1 is very akin to Theorem 10 in [13] which also generalizes Hartogs' lemma. Lelong used his Theorem 10 in [13] to prove a generalization of the Hartogs' theorem on separate analyticity. Theorem 1 will be used to give a result concerning separate analyticity of functions defined on lower dimensional subsets of $C^n$ which generalizes corresponding results due to Lelong [13] and Browder [4] (see also [6]).

## 2. An expansion of functions analytic in a symmetric neighborhood of a line segment

We shall start with the following

Lemma 2.1. *Let $D$ be a simply connected domain in the complex $z$-plane symmetric with respect to a line $l$. Let $[a, b]$ be a compact interval of $l$ contained in $D$. Then there exists a unique conformal mapping $w = g(z)$ of $D$ onto an ellipse $\mathcal{E}$ with foci $-1$ and $+1$*

$$\mathcal{E} = \{w \in C: |w + \sqrt{w^2-1}| < R\} \quad (R > 1)$$

*such that the image of $[a, b]$ under $g$ is the interval $[-1, 1]$ and $g(a) = -1$, $g(b) = 1$.* [1]

[1] We take the branch of $\sqrt{w^2-1}$ (in the $w$-plane cut along the interval $[-1, 1]$) determined by the requirement: $\sqrt{w^2-1} > 0$ for $w = \operatorname{re} w > 1$.