

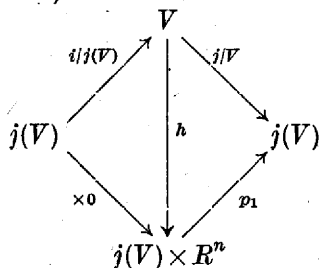
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Universal Microbundles

1. A microbundle \mathfrak{x} is a diagram

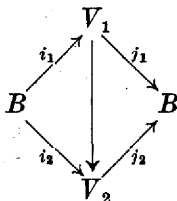
$$\mathfrak{x}: B \xrightarrow{i} E \xrightarrow{j} B$$

which consists of a base space B , a total space E , continuous maps i and j (called *injection* and *projection maps* respectively) with composition $j \circ i = \text{identity}$, such that for every point $b \in B$ there exists a neighbourhood V of $i(b)$ in E and a homeomorphism h which makes the following diagram commutative (local triviality condition)



Here $\times 0$ denotes the injection $u \rightarrow (u, 0)$ and p_1 denotes the projection $p_1(u, x) = u$. The integer n is called the *fiber dimension* of \mathfrak{x} or briefly *dimension* of \mathfrak{x} .

Let $\mathfrak{x}_\alpha: B \xrightarrow{i_\alpha} E_\alpha \xrightarrow{j_\alpha} B$ ($\alpha = 1, 2$) be two microbundles over the same base B . The microbundle \mathfrak{x}_1 is *isomorphic* to \mathfrak{x}_2 if there exist neighbourhoods V_1 of $i_1(B)$ in E_1 and V_2 of $i_2(B)$ in E_2 , and a homeomorphism $V_1 \rightarrow V_2$ such that the following diagram is commutative



The notation $\mathfrak{x}_1 \cong \mathfrak{x}_2$ will be used for this relation of isomorphism.

If $\mathfrak{x}: A \xrightarrow{i} E(\mathfrak{x}) \xrightarrow{j} A$ is a microbundle over A and $f: B \rightarrow A$ a continuous map, we define the induced microbundle $f^*\mathfrak{x}: B \xrightarrow{i'} E \xrightarrow{j'} B$ by setting

$$E \leftarrow \{(b, e) \in B \times E(\mathfrak{x}) : f(b) = j(e)\}, \quad i'(b) = (b, i \circ f(b)), \quad j'(b, e) = e.$$

If maps f and g from B to A are homotopic and B is paracompact then $f^*\mathfrak{x} \cong g^*\mathfrak{x}$.

It is well known that if the following diagram of continuous maps is commutative

$$\begin{array}{ccc} E(\mathfrak{x}_1) & \xrightarrow{F} & E(\mathfrak{x}_2) \\ j_1 \downarrow \uparrow i_1 & & i_2 \uparrow \downarrow j_2 \\ B(\mathfrak{x}_1) & \xrightarrow{f} & B(\mathfrak{x}_2) \end{array}$$

and F maps each fiber $j_1^{-1}(b)$ in one-to-one fashion into the fiber $j_2^{-1}(f(b))$ then $\mathfrak{x}_1 \cong f^*\mathfrak{x}_2$,

If $A \subset B$, then $\mathfrak{x}|_A$ denotes the microbundle $A \xrightarrow{i|_A} j^{-1}(A) \xrightarrow{j|_{j^{-1}(A)}} A$ and is called the *restriction* of the microbundle \mathfrak{x} to A .

The purpose of the present paper is the construction of a family of microbundles

$$\Gamma_p^n: Q_{p,n} \xrightarrow{i} E(\Gamma_p^n) \xrightarrow{j} Q_{p,n} \quad (p, n = 1, 2, \dots)$$

having the following universality property.

Universal microbundles theorem. *Let B be a neighbourhood retract in a Euclidean space and let $\dim_{\text{top}} B < p-1$. Then each n -dimensional microbundle over B is isomorphic to the induced microbundle $g^*\Gamma_p^n$ where $g: B \rightarrow Q_{p,n}$ is a suitably chosen continuous map.*

By Hanner theorem [1], every k -dimensional separable metric topological manifold is a neighbourhood retract in R^{2k+1} so that by applying our theorem we obtain the following

Corollary. *Let \mathfrak{x} be an n -dimensional microbundle over a k -dimensional separable metric topological manifold M . Then there exists a continuous map $g: M \rightarrow Q_{k+1,n}$ such that $\mathfrak{x} \cong g^*\Gamma_{k+1}^n$.*

Let X be a fixed topological space. We consider the semigroup $(\text{Top } X, \oplus)$ of all microbundles over X with the Whitney sum as semigroup operation. Define the following equivalence relation on $\text{Top } X$: two microbundles \mathfrak{x} and \mathfrak{x}' over X belong to the same s -class if there exist integers m, n such that the Whitney sum $\mathfrak{x} \oplus e^m$ is isomorphic to $\mathfrak{x}' \oplus e^n$ (here e^m denotes the trivial microbundle over X). J. Milnor has proved [5] that if X is a finite dimensional complex then the s -classes of microbundles over X form an Abelian group with respect to Whitney sum operation. This group will be denoted by $k_{\text{Top}} X$. The Milnor's groups $k_{\text{Top}} X$ are very important in differential topology. If X is contractible, then $k_{\text{Top}} X$ is trivial. Otherwise very little is known about those groups. For instance it is not known what is the cardinality of the very im-

portant group $k_{\text{Top}} S^n$. It is even not known whether they are finite, countable infinite or uncountable infinite. Let us suggest by the way an approach to this problem. Let $\text{card } A$ denote the cardinal number of a set A , and $[X, Y]$ the set of homotopy equivalence classes of continuous maps from X to Y . Then it is easy to verify that the following theorem holds true.

Theorem. *Let X be any n -dimensional complex. Then the following estimates for cardinal numbers are true:*

$$\text{card } k_{\text{Top}} X < \text{card } \text{Top } X < \text{card } \left(\bigcup_{k=1}^{\infty} [X, Q_{n+1,k}] \right).$$

Conjecture. If X is a compact n -dimensional complex then each set of homotopy equivalence class $[X, Q_{n+1,k}]$ ($k = 1, 2, \dots$) is at most countable.

If we could prove this, we would get a proof that $\text{Top } X$ and, in consequence, $k_{\text{Top}} X$ are at most countable.

Our universal microbundle has base $Q_{n,n}$ which is not in the class of the B 's for which it is supposed to be universal. According to the opinion of Professor S. Eilenberg the universality theorems are usually interesting only when the representing object is in the same class as the objects for which it is universal. The author is greatly indebted to Professor S. Eilenberg who has read this paper.

2. Before starting the proof of the universal microbundle theorem it is necessary to state several lemmas.

Denote by $\Omega = \Omega(n)$ the space of all continuous maps R^n into R^n with the compact-open topology. Let $\mathcal{K} = \mathcal{K}(n)$ be the subspace of all homeomorphisms which preserve the origin 0 and let $\mathcal{L} = \mathcal{L}(n) = \mathcal{K} \cup \{0\}$ be the space \mathcal{K} with attached zero map. \mathcal{K} and \mathcal{L} are to be meant as topological subspaces of Ω . Clearly \mathcal{K} is an open subset of \mathcal{L} . For $g \in \mathcal{L}$, $\varepsilon > 0$ and a compact set $K \subset R^n$ define

$$V(g, K, \varepsilon) = \{f \in \mathcal{L} : \varrho(f(x), g(x)) < \varepsilon \text{ for } x \in K\}.$$

The collection of all such $V(g, K, \varepsilon)$ is a basis for \mathcal{L} .

It is well known ([3], [2]) that the following lemma holds true:

Lemma 1. *The mappings*

- (1) $\mathcal{L} \times \mathcal{L} \ni (f, g) \rightarrow f \circ g \in \mathcal{L}$
- (2) $\mathcal{K} \ni f \rightarrow f^{-1} \in \mathcal{K}$
- (3) $\mathcal{L} \times R^n \ni (f, x) \rightarrow f(x) \in R^n$

are continuous.

Lemma 2. *If $T: B \times R^n \rightarrow R^n$ is a continuous map and if for each $b \in B$ the map*

$$T_b: R^n \ni x \rightarrow T(b, x) \in R^n$$

belongs to \mathcal{L} , then the map

$$B \ni b \rightarrow T_b \in \mathcal{L}$$

is continuous.

Proof is easily supplied.

Denote

$$\mathcal{M}_p = \underbrace{\mathfrak{L} \times \dots \times \mathfrak{L}}_p \setminus \{(0, \dots, 0)\}$$

with the product topology. Define in \mathcal{M}_p an equivalence relation S as follows: f and g in \mathcal{M}_p are S -equivalent if and only if $f = g \circ h$ for some $h \in \mathcal{H}$ (i.e., if $f = (f_1, \dots, f_p)$ and $g = (g_1, \dots, g_p)$ then $f_i = g_i \circ h$ ($i = 1, \dots, p$)).

Denote by $Q_{p,n} = \mathcal{M}_p/S$ ($p, n = 1, 2, \dots$) the quotient spaces with quotient topology and let

$$\langle \rangle : \mathcal{M}_p \ni f \rightarrow \langle f \rangle \in Q_{p,n}$$

be the quotient map.

Let $U_k = \mathfrak{L} \times \dots \times \mathcal{H} \times \dots \times \mathfrak{L}$ be the subset of \mathcal{M}_p and $V_k = \langle \rangle(U_k)$. The set V_k is open in $Q_{p,n}$ because $\langle \rangle^{-1}(V_k) = U_k$ and U_k is open in \mathcal{M}_p . Obviously

$$\mathcal{M}_p = \bigcup_{k=1}^p U_k \quad \text{and} \quad Q_{p,n} = \bigcup_{k=1}^p V_k.$$

If topological spaces (X, x_0) , $(R^n, 0)$ are homeomorphic then a continuous, non constant mapping

$$f: (X, x_0) \rightarrow (R^n)^p, 0$$

such that $p_i \circ f(X, x_0) \rightarrow (R^n, 0)$ (where $p_i: (R^n)^p \ni (a_1, \dots, a_p) \rightarrow a_i \in R^n$) is homeomorphic or constant, determines uniquely an element in $Q_{p,n}$ namely $\langle f \circ h \rangle$, where $h: (R^n, 0) \rightarrow (X, x_0)$ is homeomorphism.

Notice that if $\langle f \rangle = \langle g \rangle$ then $\text{range } f = \text{range } g$.

Let

$$E(\Gamma_p^n) = \{(\langle f \rangle, x) \in Q_{p,n} \times R^{np} : x \in \text{range } f\}$$

and

$$i: Q_{p,n} \ni \langle f \rangle \rightarrow (\langle f \rangle, 0) \in E(\Gamma_p^n)$$

$$j: E(\Gamma_p^n) \ni (\langle f \rangle, x) \rightarrow \langle f \rangle \in Q_{p,n}$$

Lemma 3. *The diagram*

$$\Gamma_p^n: Q_{p,n} \xrightarrow{i} E(\Gamma_p^n) \xrightarrow{j} Q_{p,n}$$

is an n -dimensional microbundle.

Proof. We verify local triviality over $V_k \subset Q_{p,n}$ ($k = 1, \dots, p$). Let

$$h_k: j^{-1}(V_k) \ni (\langle f \rangle, z_1, \dots, z_p) \rightarrow (\langle f \rangle, z_k) \in V_k \times R^n$$

and

$$l_k: V_k \times R^n \ni (\langle f \rangle, z) \rightarrow (\langle f \rangle, \psi_k(\varphi_k(f))(z)) \in j^{-1}(V_k)$$

where

$$\varphi_k: \underbrace{\mathfrak{L} \times \dots \times \mathfrak{L}}_{p-1} \ni (f_1, \dots, f_{p-i}) \rightarrow (f_1, \dots, e_{R^n}^{(k)}, \dots, f_{p-1}) \in U_k$$

$$\varphi_k: U_k \ni (g_1, \dots, g_p) \rightarrow (g_1 \circ g_k^{-1}, \dots, g_{k-1} \circ g_k^{-1}, g_{k+1} \circ g_k^{-1}, \dots, g_p \circ g_k^{-1}) \in \underbrace{\mathfrak{L} \times \dots \times \mathfrak{L}}_{p-1}$$

Clearly $h_k \circ l_k$ is the identity map of $V_k \times R^n$ and

$$\begin{aligned} l_k \circ h_k(\langle f \rangle, z_1, \dots, z_p) &= l_k(\langle f \rangle, z_k) = \\ &= (\langle f \rangle, f_1 \circ f_k^{-1}(z_k), \dots, z_k^{(k)}, \dots, f_p \circ f_k^{-1}(z_k)) = (\langle f \rangle, z_1, \dots, z_p). \end{aligned}$$

Since if $(z_1, \dots, z_p) \in \text{range } f$, $f = (f_1, \dots, f_p)$ then $f_j \circ f_k^{-1}(z_k) = z_j$ ($j = 1, \dots, p$). Hence $h_k = l_k^{-1}$.

The continuity of h_k is obvious, and the continuity of l_k is, by Lemma 1, an immediate consequence of the continuity of the mapping

$$U_k \times R^n \ni (f, z) \rightarrow \varphi_k \circ \varphi_k(f)(z) \in R^{np}$$

Therefore h_k is a homeomorphism. By the commutativity of the diagram

$$\begin{array}{ccc} & j^{-1}(V_k) & \\ i \nearrow & \downarrow h_k & \searrow j \\ V_k & & V_k \\ \times 0 \searrow & & \nearrow p_1 \\ & V_k \times R^n & \end{array}$$

the proof of Lemma 3 is completed.

A microbundle $\alpha: B \xrightarrow{i} E \xrightarrow{j} B$, having fiber dimension n , is said to admit a bundle if there is an open neighbourhood E_1 of $i(B)$ in E such that $j|_{E_1}: E_1 \rightarrow B$ is a fibre bundle with fibre R^n and structural group \mathcal{C} ([6]).

The fibre bundle in this case will be called an admissible bundle for α .

We will need the following

Kister-Mazur theorem. *If B is a neighbourhood retract in a Euclidean space then any microbundle over B admits a unique bundle.*

3. In the proof of the universal microbundle theorem, without loss of generality, we may assume, using the Kister-Mazur theorem, that α is an admissible bundle. Cover B by p open sets W_1, \dots, W_p with $\alpha|_{W_i}$ trivial. (This is possible by the fundamental theorem of dimension theory). This means that the following p diagrams are commutative

$$\begin{array}{ccc} & j^{-1}(W_k) & \\ i \nearrow & \downarrow \tilde{f}_k & \searrow j \\ W_k & & W_k \\ \alpha_0 \searrow & & \nearrow p_1 \\ & W_k \times R^n & \end{array}$$

and \tilde{f}_k ($k = 1, \dots, p$) are homeomorphisms.

Let $\varphi_1, \dots, \varphi_p$ be a partition of unity with the support $\varphi_i \subset W_i$ for each i . We define continuous maps $f_k: E(\alpha) \rightarrow R^n$ ($k = 1, \dots, p$) by the formula:

$$f_k(e) = \begin{cases} 0 & \text{if } j(e) \notin W_k \\ \varphi_k(j(e)) \cdot p \circ \tilde{f}_k(e) & \text{if } j(e) \in W_k \end{cases}$$

where $p: B \times R^n \rightarrow R^n$ is the natural projection and $j: E(\alpha) \rightarrow B$ denotes the projection maps in the microbundle α .

Define

$$F: E(\mathfrak{x}) \ni e \rightarrow (f_1(e), \dots, f_p(e)) \in R^{np}.$$

Clearly F is a continuous map.

The restriction of F to the fiber of $b \in B$ determines, according to our remark, an element in $Q_{p,n}$, which will be denoted by $\langle F_b \rangle$.

Let

$$G: E(\mathfrak{x}) \ni e \rightarrow (\langle F_{j(e)} \rangle, \hat{F}(e)) \in E(\Gamma_p^n).$$

and set $g = j \circ G \circ i$, where i is the injection map of \mathfrak{x} and j projection map of Γ_p^n .

It is easy to verify that G maps each fibre of \mathfrak{x} in one-to-one fashion into some fibre of Γ_p^n . The proof of theorem will be completed, if we show that G is continuous. But the continuity of G follows from the fact that each maps

$$G \circ (\tilde{f}_k)^{-1}: W_k \times R^n \rightarrow Q_{p,n} \times R^{np} \quad (k = 1, \dots, p)$$

is continuous, by Lemma 2.

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