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Formal Solutions of a Functional Equation

In our papers [3] and [4], dealing with the existence of analytic solutions of the functional equation

$$\varphi(z) = h(z, \varphi[f(z)]) \tag{1}$$

in multidimensional spaces, we assumed that there exists a formal solution of this equation. In Section 3 of the present paper we give some conditions for the existence of a formal solution of equation (1). The first two sections contain some preliminary results from the theory of multidimensional matrices and formal series.

1. **Multidimensional matrices.** Let us fix a sequence of positive integers $I_1, \ldots, I_\mu$. We introduce in the set of sequences $(i_1, \ldots, i_\mu)$, $i_1 \leqslant I_1, \ldots, i_\mu \leqslant I_\mu$, an order by first difference:

$$(i_1, i_2) \precsim (i_1', i_2') \Longleftrightarrow [i_1 < i_1' \vee (i_1 = i_1', i_2 < i_2')]\,,$$

$$(i_1, \ldots, i_\mu) \precsim (i_1', \ldots, i_\mu') \Longleftrightarrow \{i_1 < i_1' \vee [i_1 = i_1', (i_2, \ldots, i_\mu) \precsim (i_2', \ldots, i_\mu')]\}\,.$$

The above relation orders the set of sequences $(i_1, \ldots, i_\mu)$, $i_1 \leqslant I_1, \ldots, i_\mu \leqslant I_\mu$. Thus we have established a one-to-one correspondence between sequences $(i_1, \ldots, i_\mu)$ and the positive integers up to $I_1 I_2 \ldots I_\mu$. The number corresponding to a sequence $(i_1, \ldots, i_\mu)$ will be called the *index* of this sequence and denoted by $\mathrm{ind}(i_1, \ldots, i_\mu)$.

The purpose of introducing the index is a possibility of passing from multidimensional matrices to matrices with a smaller number of indices. A similar idea may be found in the notion of the curvature deffinor introduced by A. Jakubowicz [2].

For two-term sequences (i, j), $i \leqslant I$, $j \leqslant J$, the index may be calculated from the formula

$$\operatorname{ind}(i, j) = (i-1)J + j .$$

The reader will verify that the definition of the index given above is equivalent to the following, recurrent one

$$\operatorname{ind}(i_1, \ldots, i_\mu) = \operatorname{ind}[\operatorname{ind}(i_1, \ldots, i_{\mu-1}), i_\mu] . \tag{2}$$

Lemma 1. *For an arbitrary positive integer μ, $\mu \geqslant 3$,*

$$\operatorname{ind}[\operatorname{ind}(i_1, \ldots, i_{\mu-1}), i_\mu] = \operatorname{ind}[i_1, \operatorname{ind}(i_2, \ldots, i_\mu]) .$$

Proof. We have for $\mu = 3$

$$\begin{aligned}\operatorname{ind}[\operatorname{ind}(i_1, i_2), i_3] &= [\operatorname{ind}(i_1, i_2)-1]I_3 + i_3 = [(i_1-1)I_2 + (i_2-1)]I_3 + i_3 \\ &= (i_1-1)I_2I_3 + (i_2-1)I_3 + i_3 = (i_1-1)I_2I_3 + \operatorname{ind}(i_2, i_3) .\end{aligned}$$

Thus (let us note that ind (i_2, i_3) ranges from 1 to I_2I_3)

$$\operatorname{ind}[\operatorname{ind}(i_1, i_2), i_3] = \operatorname{ind}[i_1, \operatorname{ind}(i_2, i_3)] . \tag{3}$$

We assume that for a $q \geqslant 4$ the equality

$$\operatorname{ind}[\operatorname{ind}(i_1, \ldots, i_{q-2}), i_{q-1}] = \operatorname{ind}[i_1, \operatorname{ind}(i_2, \ldots, i_{q-1})]$$

holds. Then

$$\begin{aligned}\operatorname{ind}[\operatorname{ind}(i_1, \ldots, i_{q-1}), i_q] &= \operatorname{ind}\{\operatorname{ind}[\operatorname{ind}(i_1, \ldots, i_{q-2}), i_{q-1}], i_q\} \\ &= \operatorname{ind}\{\operatorname{ind}[i_1, \operatorname{ind}(i_2, \ldots, i_{q-1})], i_q\} .\end{aligned}$$

Hence and by (3) we obtain

$$\begin{aligned}\operatorname{ind}[\operatorname{ind}(i_1, \ldots, i_{q-1}), i_q] &= \operatorname{ind}\{i_1, \operatorname{ind}[\operatorname{ind}(i_2, \ldots, i_{q-1}), i_q]\} \\ &= \operatorname{ind}[i_1, \operatorname{ind}(i_2, \ldots, i_q)] ,\end{aligned}$$

which was to be proved.

Let C denote the space of complex numbers, and let $I_1, \ldots, I_\mu$; $K_1, \ldots, K_\nu$ denote arbitrarily fixed positive integers. To every sequence $(i_1, \ldots, i_\mu; k_1, \ldots, k_\nu)$, $i_1 \leqslant I_1, \ldots, i_\mu \leqslant I_\mu$, $k_1 \leqslant K_1, \ldots, k_\nu \leqslant K_\nu$ we assign a complex number $a^{i_1\ldots i_\mu}_{k_1\ldots k_\nu}$. We say that we have defined a matrix

$$a = [a^{i_1\ldots i_\mu}_{k_1\ldots k_\nu}]_{i_1\leqslant I_1, \ldots, i_\mu\leqslant I_\mu, k_1\leqslant K_1, \ldots, k_\nu\leqslant K_\nu}$$

of type (valence) (μ, ν). The multiplication of a matrix a by a complex number α and the addition of two matrices a and b of the same type are defined by the formulas

$$\alpha a = [\alpha a^{i_1\ldots i_\mu}_{k_1\ldots k_\nu}]_{i_1\leqslant I_1,\ldots,i_\mu\leqslant I_\mu,k_1\leqslant K_1,\ldots,k_\nu\leqslant K_\nu} ,$$

$$a + b = [a^{i_1\ldots i_\mu}_{k_1\ldots k_\nu} + b^{i_1\ldots i_\mu}_{k_1\ldots k_\nu}]_{i_1\leqslant I_1\ldots,i_\mu\leqslant I_\mu,k_1\leqslant K_1,\ldots,k_\nu\leqslant K_\nu} .$$

The set of all matrices of a given type with operations defined in such a manner forms a vector space.

The *tensor product* of

$$a = [a^{i_1\ldots i_\mu}_{j_1\ldots j_\nu}]_{i_1\leqslant I_1,\ldots,i_\mu\leqslant I_\mu, j_1\leqslant J_1,\ldots,j_\nu\leqslant J_\nu}$$

and

$$b = [b^{k_1\ldots k_\varrho}_{l_1\ldots l_\sigma}]_{k_1\leqslant K_1,\ldots,k_\varrho\leqslant K_\varrho, l_1\leqslant L_1,\ldots,l_\sigma\leqslant L_\sigma}$$

is the matrix

$$a\times b = [c^{i_1\ldots i_\mu k_1\ldots k_\varrho}_{j_1\ldots j_\nu l_1\ldots l_\sigma}]_{i_1\leqslant I_1,\ldots,i_\mu\leqslant I_\mu, k_1\leqslant K_1,\ldots, k_\varrho\leqslant K_\varrho, j_1\leqslant J_1,\ldots,j_\nu\leqslant J_\nu, l_1\leqslant L_1,\ldots,l_\sigma\leqslant L_\sigma},$$

where $c^{i_1\ldots i_\mu k_1\ldots k_\varrho}_{j_1\ldots j_\nu l_1\ldots l_\sigma} = a^{i_1\ldots i_\mu}_{j_1\ldots j_\nu} b^{k_1\ldots k_\varrho}_{l_1\ldots l_\sigma}$.

The tensor product is distributive with respect to the addition of matrices of the same type, and associative.

The matrix

$$ab = \Big[\sum_{j_1\leqslant J_1,\ldots,j_\nu\leqslant J_\nu} a^{i_1\ldots i_\mu}_{j_1\ldots j_\nu} b^{j_1\ldots j_\nu}_{k_1\ldots k_\lambda}\Big]_{i_1\leqslant I_1,\ldots,i_\mu\leqslant I_\mu, k_1\leqslant K_1,\ldots k_\lambda\leqslant K_\lambda}$$

is called the *matrix product* of $a = [a^{i_1\ldots i_\mu}_{j_1\ldots j_\nu}]_{i_1\leqslant I_1,\ldots,i_\mu\leqslant I_\mu, j_1\leqslant J_1,\ldots,j_\nu\leqslant J_\nu}$ and $b = [b^{j_1\ldots j_\nu}_{k_1\ldots k_\lambda}]_{j_1\leqslant J_1,\ldots,j_\nu\leqslant J_\nu, k_1\leqslant K_1,\ldots,k_\lambda\leqslant K_\lambda}$.

The matrix product is distributive with respect to the addition of matrices of the same type.

We define the *flat product* $a*b$ of matrices $a = [a^{i_1}_{i_2}]_{i_1,i_2\leqslant J}$ and $b = [b^{j_1}_{j_2}]_{j_1,j_2\leqslant J}$ by the formula

$$a*b = [c^{k_1}_{k_2}]_{k_1,k_2\leqslant IJ},\ c^{k_1}_{k_2} = a^{i_1}_{i_2} b^{j_1}_{j_2},$$

where $k_1 = \operatorname{ind}(i_1, j_1)$, $k_2 = \operatorname{ind}(i_2, j_2)$.

The flat product is associative. In fact, we have according to the definition

$$a*(b*c) = a*[d^{l_1}_{l_2}]_{l_1,l_2\leqslant JK} = [e^{m_1}_{m_2}]_{m_1,m_2\leqslant IJK},$$

where $d^{l_1}_{l_2} = b^{j_1}_{j_2} c^{k_1}_{k_2}$, $e^{m_1}_{m_2} = a^{i_1}_{i_2} d^{l_1}_{l_2}$, $l_1 = \operatorname{ind}(j_1, k_1)$, $l_2 = \operatorname{ind}(j_2, k_2)$ and $m_1 = \operatorname{ind}(i_1, l_1)$, $m_2 = \operatorname{ind}(i_2, l_2)$. It follows from Lemma 1 that $m_1 = \operatorname{ind}(n_1, k_1)$, $m_2 = \operatorname{ind}(n_2, k_2)$, where $n_1 = \operatorname{ind}(i_1, j_1)$, $n_2 = \operatorname{ind}(i_2, j_2)$, whence

$$a*(b*c) = [f^{n_1}_{n_2}]_{n_1,n_2\leqslant IJ} * c,$$

where $f^{n_1}_{n_2} = a^{i_1}_{i_2} b^{j_1}_{j_2}$. Thus

$$a*(b*c) = (a*b)*c.$$

Lemma 2. *If for* $i_1 < i_2$ $a^{i_1}_{i_2} = 0$ *and for* $j_1 < j_2$ $b^{j_1}_{j_2} = 0$, *then for* $k_1 < k_2$ $c^{k_1}_{k_2} = 0$, *where* $[c^{k_1}_{k_2}]_{k_1,k_2\leqslant IJ} = a*b$. *Moreover* $c^k_k = a^i_i b^j_j$, *where* $k = \operatorname{ind}(i, j)$.

Proof. $k_1 < k_2 \Leftrightarrow \operatorname{ind}(i_1, j_1) < \operatorname{ind}(i_2, j_2) \Leftrightarrow [i_1 < i_2 \vee (i_1 = i_2, j_1 < j_2)] \Rightarrow a^{i_1}_{i_2} = 0 \vee b^{j_1}_{j_2} = 0 \Rightarrow c^{k_1}_{k_2} = 0$. If $k_1 = k_2$ then $i_1 = i_2$, $j_1 = j_2$ and $c^k_k = a^i_i b^j_j$.

2. Formal power series. Let us fix positive integers m and n. Let $\{a_\mu\}$ be a sequence of matrices of type $(1, \mu)$ $[a^k_{i_1 \dots i_\mu}]_{k \leqslant m, i_1, \dots, i_\mu \leqslant n}$. We put

$$z^\mu = \underbrace{z \times \dots \times z}_{\mu \text{ times}},$$

where $z = (z^1, \dots, z^n)$, $z^i \in C$. $a_\mu z^\mu$ denote the matrix product of a_μ and z^μ, and thus it is a point of the complex m-space: $a_\mu z^\mu \in C^m$. The symbol

$$A(z) = \sum_{\mu=1}^{\infty} a_\mu z^\mu \tag{4}$$

is called a *formal power series.*

Let us note that the terms of series (4) are in fact elements of the m-dimensional complex space.

Series (4) and $A'(z) = \sum\limits_{\mu=1}^{\infty} a'_\mu z^\mu$ are equal iff $a_\mu = a'_\mu$.

If $B(z) = \sum\limits_{\mu=1}^{\infty} b_\mu z^\mu$, where $b_\mu = [b^i_{j_1 \dots j_\mu}]_{i, j_1, \dots, j_\mu \leqslant n}$, then the series *)

$$A[B(z)] \overset{\mathrm{df}}{=} \sum_{\mu=1}^{\infty} \Big[\sum_{\nu=1}^{\mu} \sum_{\substack{\varrho_1 \dots \varrho_\nu \in N \\ \varrho_1 + \dots + \varrho_\nu = \mu}} a_\mu (b_{\varrho_1} \times \dots \times b_{\varrho_\nu}) \Big] z^\mu \tag{5}$$

is called the *superposition* of A and B.

It is possible to prove formula (5) developping a theory of formal power series in several variables analogous to that given in [1] in the case of a single variable. Since in the present paper we aim at solving equation (1), we cannot go very deep into this theory and we assume formula (5) as definition. The same remark applies also to formulas (7), (8), (9).

Let $\{a_{\mu\nu}\}$ be a sequence of matrices $[a^k_{j_1 \dots j_\mu l_1 \dots l_\nu}]_{k \leqslant m, j_1, \dots, j_\mu \leqslant n, l_1, \dots, l_\nu \leqslant m}$. Let $z = (z^1, \dots, z^n)$, $w = (w^1, \dots, w^m)$, $z^i, w^k \in C$.
The symbol

$$A(z, w) = \sum_{\substack{\mu, \nu = 0 \\ \mu + \nu \geqslant 1}}^{\infty} a_{\mu\nu} z^\mu \times w^\nu \tag{6}$$

is called a *formal power series in two variables.*

If $C(z) = \sum\limits_{\nu=1}^{\infty} c_\nu z^\nu$, $c_\nu = [c^k_{i_1 \dots i_\nu}]_{k \leqslant m, i_1, \dots, i_\nu \leqslant n}$, then

$$A[z, C(z)] \overset{\mathrm{df}}{=} \sum_{\mu=1}^{\infty} \Big[a_{\mu 0} + \sum_{\nu=1}^{\mu} \sum_{\lambda=0}^{\mu-1} a_{\lambda\nu} \cdot \Big(\sum_{\substack{\varrho_1 \dots \varrho_\nu \in N \\ \varrho_1 + \dots + \varrho_\nu = \mu - \lambda}} c_{\varrho_1} \times \dots \times c_{\varrho_\nu} \Big) \Big] z^\mu . \tag{7}$$

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*) N denotes the set of all positive integers.

If $C(w) = cw$, where $c = [c_k^l]_{l,k\leqslant m}$, then

$$C[A(z, w)] \stackrel{\text{df}}{=} \sum_{\substack{\mu,\nu=0\\ \mu+\nu\geqslant 1}}^{\infty} (ca_{\mu\nu}) z^{\mu} \times w^{\nu} . \tag{8}$$

If $B(z) = bz$ and $C(w) = cw$, $b = [b_j^i]_{i,j\leqslant n}$, $c = [c_l^k]_{k,l\leqslant m}$, then

$$A[B(z), C(z)] \stackrel{\text{df}}{=} \sum_{\substack{\mu,\nu=0\\ \mu+\nu\geqslant 1}}^{\infty} [a_{\mu\nu}(\underbrace{b\times \ldots \times b}_{\mu \text{ times}}) \times (\underbrace{c\times \ldots \times c}_{\nu \text{ times}})](z^{\mu} \times w^{\nu}) . \tag{9}$$

3. **A formal solution of equation (1).** Suppose that we are given formal series

$$h(z, w) = \sum_{\substack{\mu,\nu=0\\ \mu+\nu\geqslant 1}}^{\infty} a_{\mu\nu}(z^{\mu} \times w^{\nu})$$

and

$$f(z) = \sum_{\mu=1}^{\infty} b_{\mu} z^{\mu}, \quad b = [b^i_{j_1\ldots j_\mu}]_{i,j_1,\ldots,j_\mu\leqslant n} .$$

We are seeking a series

$$\varphi(z) = \sum_{\mu=1}^{\infty} c_{\mu} z^{\mu}, \qquad c_{\mu} = [c^k_{i_1\ldots i_\mu}]_{k\leqslant m, i_1,\ldots,i_\mu\leqslant n} \tag{10}$$

fulfilling equation (1). By this we mean that if we form the superposition $\varphi[f(z)]$ according to formula (5), and next the superposition $h(z, \varphi[f(z)])$ according to formula (7), then the formal power series obtained will be equal to (10).

In virtue of formula (5) we get

$$\varphi[f(z)] = \sum_{\mu=1}^{\infty} \Big[\sum_{\nu=1}^{\mu} \sum_{\substack{\varrho_1\ldots\varrho_\nu \in N\\ \varrho_1+\ldots+\varrho_\nu=\mu}} c_{\nu}(b_{\varrho_1} \times \ldots \times b_{\varrho_\nu})\Big] z^{\mu} .$$

Hence and from (7) we have

$$\begin{aligned} (11)\quad & h(z, \varphi[f(z)]) \\ &= \sum_{\mu=1}^{\infty} \Big\{ a_{\mu 0} + \sum_{\nu=1} \sum_{\lambda=0} a_{\lambda\nu} \cdot \Big(\sum_{\substack{\varrho_1\ldots\varrho_\nu \in N\\ \varrho_1+\ldots+\varrho_\nu=\mu-\lambda}} \Big[\sum_{\tau=1}^{\varrho_1} \sum_{\substack{\gamma_1\ldots\gamma_\tau \in N\\ \gamma_1+\ldots+\gamma_\tau=\varrho_1}} c_{\tau}(b_{\gamma_1} \times \ldots \times b_{\gamma_\tau})\Big] \times \\ &\qquad \times \ldots \times \Big[\sum_{\tau=1}^{\varrho_\nu} \sum_{\substack{\gamma_1\ldots\gamma_\tau \in N\\ \gamma_1+\ldots+\gamma_\tau=\varrho_\nu}} c_{\tau}(b_{\gamma_1} \times \ldots \times b_{\gamma_\tau})\Big]\Big)\Big\} z^{\mu} . \end{aligned}$$

Formal series (10) fulfils equation (1) if and only if the coefficients of z^μ in series (10) and (11) are equal, i.e.

$$a_{\mu 0}+\sum_{\nu=1}^{\mu}\sum_{\lambda=0}^{\mu-1} a_{\lambda\nu}\cdot\Big(\sum_{\substack{\varrho_1\ldots\varrho_\nu\in N\\ \varrho_1+\ldots+\varrho_\nu=\mu-\lambda}}\Big[\sum_{\tau=1}^{\varrho_1}\sum_{\substack{\gamma_1\ldots\gamma_\tau\in N\\ \gamma_1+\ldots+\gamma_\tau=\varrho_1}} c_\tau\cdot(b_{\gamma_1}\times\ldots\times b_{\gamma_\tau})\Big]\times$$

$$\times\ldots\times\Big[\sum_{\tau=1}^{\varrho_\nu}\sum_{\substack{\gamma_1\ldots\gamma_\tau\in N\\ \gamma_1+\ldots+\gamma_\tau=\varrho_\nu}} c_\tau\cdot(b_{\gamma_1}\times\ldots\times b_{\gamma_\tau})\Big]\Big)=c_\mu .$$

In virtue of the distributivity of the operations $\cdot$ and $\times$ with respect to the addition this may be written in the form

(12) $$a_{\mu 0}+\sum_{\nu=1}^{\mu}\sum_{\lambda=0}^{\mu-1}\sum_{\substack{\varrho_1\ldots\varrho_\nu\in N\\ 1+\ldots+=-}}\sum_{\tau_1=1}^{\varrho_1}\ldots\sum_{\tau_\nu=1}^{\varrho_\nu}\sum_{\substack{\gamma_1^1\ldots\gamma_{\tau_1}^1\in N\\ \gamma_1^1+\ldots+\gamma_{\tau_1}^1=\varrho_1}}\ldots$$

$$\sum_{\substack{\gamma_1^\nu\ldots\gamma_{\tau_\nu}^\nu\in N\\ \gamma_1^\nu+\ldots+\gamma_{\tau_\nu}^\nu=\varrho_\nu}} a_{\lambda\nu}\cdot\{[c_{\tau_1}\cdot(b_{\gamma_1^1}\times\ldots\times b_{\gamma_{\tau_1}^1})]\times\ldots\times[c_{\tau_\nu}\cdot(b_{\gamma_1^\nu}\times\ldots\times b_{\gamma_{\tau_\nu}^\nu})]\}=c_\mu .$$

Let $\tau_1=\mu$. Since $\tau_1\leqslant\varrho_1\leqslant\mu-\lambda$, $\mu-\lambda\geqslant\mu$. The last inequality occurs only if $\lambda=0$. So $\mu=\varrho_1$, whence $\nu=1$ and the expression under the Σ signs on the left-hand side of (12) reduces to

(13) $$a_{\mu 0}+\sum_{\substack{\gamma_1\ldots\gamma_\mu\in N\\ \gamma_1+\ldots+\gamma_\mu=\mu}} a_{01}\cdot[c_\mu\cdot(b_{\gamma_1}\times\ldots\times b_{\gamma_\mu})] .$$

(Thus (13) represents those summands in (12) that contain c_μ). However, $\gamma_1+\ldots+\gamma_\mu=\mu$ implies $\gamma_1=\ldots=\gamma_\mu=1$. Thus, we get from (12)

$$a(c_1,\ldots,c_{\mu-1})+a_{01}[c_\mu\cdot\underbrace{(b_1\times\ldots\times b_1)}_{\mu\text{ times}}]=c_\mu ,$$

where $a(c_1,\ldots,c_{\mu-1})$ stands for all the summands that do not contain c_μ. The last equality can be written in the form

(14) $$[a^k_{i_1\ldots i_\mu}]_{k\leqslant m,i_1,\ldots,i_\mu\leqslant n}+$$

$$+\Big[\sum_{l=1}^{m}a^k_{0l}\sum_{j_1\ldots j_\mu=1}^{n}c^l_{j_1\ldots j_\mu}b^{j_1}_{i_1}\ldots b^{j_\mu}_{i_\mu}\Big]_{k\leqslant m,i_1,\ldots,i_\mu\leqslant n}=[c^k_{i_1\ldots i_\mu}]_{k\leqslant m,i_1,\ldots,i_\mu\leqslant n} .$$

We write $\bar a_{ki_1\ldots i_\mu}=a^k_{i_1\ldots i_\mu}$, $\bar c_{lj_1\ldots j_\mu}=c^l_{j_1\ldots j_\mu}$ and $\bar a^l_k=a^k_{0l}$. Then (14) is equivalent to

$$[\bar a_{ki_1\ldots i_\mu}]_{k\leqslant m,i_1,\ldots,i_\mu\leqslant n}+$$

$$+\Big[\sum_{l=1}^{m}\sum_{j_1\ldots j_\mu=1}^{n}\bar c_{lj_1\ldots j_\mu}\bar a^l_k b^{j_1}_{i_1}\ldots b^{j_\mu}_{i_\mu}\Big]_{k\leqslant m,i_1,\ldots,i_\mu\leqslant n}=[\bar c_{ki_1\ldots i_\mu}]_{k\leqslant m,i_1,\ldots,i_\mu\leqslant n}$$

or

$$[\bar{a}_{ki_1\ldots i_\mu}]_{k\leqslant m, i_1,\ldots,i_\mu\leqslant n} + $$
$$+ [\bar{c}_{lj_1\ldots j_\mu}]_{l\leqslant m, j_1,\ldots,j_\mu\leqslant n} \cdot [\bar{a}^l_k b^{j_1}_{i_1}\ldots b^{j_\mu}_{i_\mu}]_{m,j_1,\ldots,j_\mu\leqslant n, k\leqslant m, i_1,\ldots,i_\mu\leqslant n}$$
$$= [\bar{c}_{ki_1\ldots i_\mu}]_{k\leqslant m, i_1,\ldots,i_\mu\leqslant n}\,.$$

Let us write the matrix $[\bar{a}^l_k b^{j_1}_{i_1}\ldots b^{j_\mu}_{i_\mu}]_{l\leqslant m, j_1,\ldots,j_\mu\leqslant n, k\leqslant m, i_1,\ldots,i_\mu\leqslant k}$ in the form of a matrix $[\beta^p_q]_{p,qm\leqslant n^\mu}$, where $\beta^p_q = a^l_k b^{j_1}_{i_1}\ldots b^{j_\mu}_{i_\mu}$, $p = \operatorname{ind}(l, j_1, \ldots j_\mu)$ and $q = \operatorname{ind}(k, i_1, \ldots, i_\mu)$. Similarly, let us write matrices $[\bar{a}_{ki_1\ldots i_\mu}]_{k\leqslant m, i_1,\ldots,i_\mu\leqslant n}$, $[\bar{c}_{ki_1\ldots i_\mu}]_{k\leqslant m, i_1,\ldots,i_\mu\leqslant n}$ as vectors (one-row matrices) $[a_q]_{q\leqslant mn^\mu}$, $[c_p]_{p\leqslant mn^\mu}$, where again $q = \operatorname{ind}(k, i_1, \ldots, i_\mu)$. It follows fro mthe definition of the flat multiplication and from its associativity that

$$[a_q]_{q\leqslant mn^\mu} + [c_q]_{q\leqslant mn^\mu}\cdot (a^T_{01} * \overset{\mu}{\overline{|*|}}\, b_1) = [c_q]_{q\leqslant mn^\mu}\,,$$

where

$$\overset{\mu}{\overline{|*|}}\, b_1 = \underbrace{b_1 * \ldots * b_1}_{\mu\ \text{times}}\,.$$

Hence

$$(15)\qquad [a_q]_{q\leqslant mn^\mu} = [c_q]_{q\leqslant mn^\mu}\cdot (E - a^T_{01} * \overset{\mu}{\overline{|*|}}\, b_1)\,.$$

The existence of a formal solution of equation (1) is equivalent to the existence of a solution of the infinite sequence of systems of linear equations (15). Let us write

$$\Delta_\mu = \det(a^T_{01} * \overset{\mu}{\overline{|*|}}\, b_1 - E)\,.$$

If for every $\mu = 1, 2, \ldots$, we have $\Delta_\mu \neq 0$, then system (15) possesses exactly one solution and consequently there exists exactly one formal series fulfilling equation (1). If, for some μ, $\Delta_\mu = 0$, then system (15) either has no solution, or it has a finite-parameter family of solutions. This depends on $[a_q]$ according to the general rules of the theory of linear equations.

Let us assume for the moment that the matrix a_{01} is triangular with zeros under the main diagonal, and the matrix b_1 is triangular with zero sover the main diagonal. The elements of the main diagonal of those matrices are their characteristic roots $\sigma_1, \ldots, \sigma_m$ and $s_1, \ldots, s_n$, respectively. It follows from Lemma 2 that the matrix $a^T_{01} * \overset{\mu}{\overline{|*|}}\, b_1$ has zeros over the main diagonal and on this diagonal all products of the form $\sigma_k s_{i_1}\ldots s_{i_\mu}$, where $k \leqslant m$, $i_1, \ldots, i_\mu \leqslant n$. Thus

$$\Delta_\mu = \prod_{k=1}^{m} \prod_{i_1\ldots i_\mu=1}^{n} (\sigma_k s_{i_1}\ldots s_{i_\mu} - 1)\,.$$

In order to be sure of the existence of a formal solution of equation (1) it is enough to assume that the condition

$$(16)\qquad \forall_{k\leqslant m} \forall_{p_1,\ldots,p_n = 0,1,2,\ldots} (\sigma_k s_1^{p_1}\ldots s_n^{p_n} \neq 1)$$

is fulfilled.