ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO

Nr CCIII PRACE MATEMATYCZNE, ZESZYT 13 1969

Wiesław Solak

On the Integrals of Furcation of an Ordinary Differential Equation

1. In the present paper the differential equation

$$y' = f(x, y) \tag{1.1}$$

will be considered. Assume that the function $f(x, y)$ is defined and continuous in a rectangle (finite or infinite)

$$\Pi = \{(x, y): a < x < b, c < y < d\}$$

and that through every point $P \in \Pi$ passes one integral of equation (1.1). By the continuity of $f(x, y)$ and uniqueness of equation (1.1), an integral issuing from an arbitrary point $P \in \Pi$ can be extended uniquely up to the boundary of Π ([1], p. 62).

The integral of equation (1.1) passing through the point P belonging to Π will be denoted by $\varphi(x, P)$. The maximum interval of existence of solution will be called its *saturated interval.*

A sequence of integrals $\{y_k\}$ is said *to condense at a point* Q if there exists a sequence of points P_k such that

$$P_k \to Q \text{ and } y_k(x) = \varphi(x, P_k) .$$

A sequence of integrals $\{y_k\}$ is said *to condense on an integral* y if that sequence condenses at every point Q on y.

Theorem A. *If a sequence of integrals $\{y_k\}$ condenses at a point Q then the sequence $\{y_k\}$ condenses on the integral $\varphi(x, Q)$* (see [2]).

Two different integrals $\bar{y}$ and $\bar{\bar{y}}$ are said to be *associated* if there exists a sequence of integrals $\{y_k\}$ that condenses on $\bar{y}$ and $\bar{\bar{y}}$ ([2]).

We say that an integral is the *integral of furcation* if it has at least one associated integral. We call an integral an *ordinary* one if it is not an integral of furcation ([2]).

A straight line $x = \beta$, $\beta \in (a, b]$ ($\beta \in [a, b)$) is called the *upper right-hand* (*upper left-hand*) *asymptote* for a function $\lambda(x)$ defined in the interval $(a, \beta) \subset \subset (a, b)$ $\big((\beta, a) \subset (a, b)\big)$ if

$$\lim_{x\to\beta-0} \lambda(x) = d \qquad \big(\lim_{x\to\beta+0} \lambda(x) = d\big).$$

Similarly, a straight line $x = \beta$, $\beta \in (a, b]$ ($\beta \in [a, b)$) is called the *lower right-hand* (*lower left-hand*) *asymptote* for a function $\lambda(x)$ if

$$\lim_{x\to\beta-0} \lambda(x) = c \qquad \big(\lim_{x\to\beta+0} \lambda(x) = c\big).$$

A straight line $x = \beta$ is the *upper right-hand asymptote of furcation* if there exist points a, γ and δ ($a < \beta$, $\beta \leqslant \gamma < \delta$) and two integrals $\lambda(x)$ and $\mu(x)$ of equation (1.1) such that

$$\begin{aligned} &y = \lambda(x) && \text{is defined for } a < x < \beta\,, \\ &y = \mu(x) && \text{is defined for } \gamma < x < \delta\,, \\ &\lim_{x\to\beta-0} \lambda(x) = d\,, && \lim_{x\to\gamma+0} \mu(x) = d \end{aligned}$$

and through the line $x = \beta$ does not pass any of the integrals of equation (1.1) having at any point of the interval $(\beta, \gamma]$ the left-hand boundary equal to d.

The upper left-hand, lower right-hand and lower left-hand asymptotes are defined accordingly.

2. Lemma 1. *If the integral* $\varphi(x, P)$, $P = (\beta, y)$, $c < y < d$, *satisfies for a certain* $\bar\beta \in (\beta, b]$ *the condition*

$$\lim_{x\to\bar\beta-0} \varphi(x, P) = d\,,$$

then every integral issuing from the segment AP, $A = (\beta, d)$, *has its right-hand boundary equal to* d *and the right-hand end of its saturated interval belongs to the interval* $(\beta, \bar\beta]$.

Theorem 1. *A necessary and sufficient condition for equation* (1.1) *to have an integral of furcation is the existence of an asymptote of furcation.*

Proof. Suppose first that there exists for equation (1.1) an integral of furcation and let $y = \lambda(x)$ ($a < x < \beta$) and $y = \mu(x)$ ($\gamma < x < \delta$) be the associated integrals. We know that $\beta \leqslant \gamma$ ([2]). By the theorem on the extension of an arbitrary solution of equation (1.1) up to the boundary, there exists the limit of the function $\lambda(x)$ when x tends to β and the limit of the function $\mu(x)$ when x tends to γ. We denote them by m and l, respectively, $\lim_{x\to\beta-0} \lambda(x) = m$, $\lim_{x\to\gamma+0} \mu(x) = l$.

We will prove that $m = l$. Assume that $m \neq l$; for example $m = d$, $l = c$. It can be easily seen that the sequence $y_k = \varphi(x, P_k)$ condensing by hypothesis on the integrals $\lambda(x)$ and $\mu(x)$ passes through the lines $x = \beta$ and $x = \gamma$ starting from a certain index N, $\varphi(\beta, P_k) \to d$, $\varphi(\gamma, P_k) \to c$ for $k \to \infty$.

From the sequence $\varphi(x, P_k)$ we choose a subsequence $\varphi(x, P_n)$ satisfying the conditions: $\varphi(\beta, P_n) < \varphi(\beta, P_{n+1})$ for every $n \geqslant N$. The sequence $\varphi(x, P_n)$ condenses on the integrals λ and μ, but $c < \varphi(\gamma, P_n) < \varphi(\gamma, P_{n+1})$ for $n \geqslant N$. Hence $\varphi(\gamma, P_n) \not\to c$ which is a contradiction. Therefore we have

$$\lim_{x \to \beta - 0} \lambda(x) = \lim_{x \to \gamma + 0} \mu(x) .$$

Assume that $m = l = d$ and that the line $x = \beta$ is not the upper right-hand asymptote of furcation. There exists an integral $\nu(x)$ defined for $x \geqslant \beta$ and such that $\lim_{x \to \varrho - 0} \nu(x) = d$, $\beta < \varrho \leqslant \gamma$. It follows easily that $\varphi(\beta, P_k) > \nu(\beta)$ for $k > k_0$. Hence, by Lemma 1, the right ends of the saturated intervals of the integrals $\varphi(x, P_k)$ for $k > k_0$ lie in $(\beta, \varrho]$ and the sequence $\varphi(x, P_k)$ cannot condense on the integral $\mu(x)$ defined for $x > \varrho$.

Suppose now that there exists for equation (1.1) an asymptote of furcation. Let the line $x = \beta$ be the upper right-hand asymptote of furcation and let a, β, γ, δ and the functions $\lambda(x), \mu(x)$ satisfy the condition of the definition of an asymptote of furcation.

We choose the sequence of integrals $\varphi(x, P_k)$ defined for $x = \beta$ and such that $\varphi(\beta, P_k) \to d$ for $k \to \infty$.

Let $\varepsilon > 0$ be arbitrary fixed so that the line $y = d - \varepsilon$ intersects the integrals $\lambda = \lambda(x)$ and $\mu = \mu(x)$. There exists such an index k_0 that, for $k \geqslant k_0$, $\varphi(\beta, P_k) > d - \varepsilon$.

For $k > k_0$ the integrals $\varphi(x, P_k)$ intersect the line $y = d - \varepsilon$ from both sides of the straight line $x = \beta$. We denote by x_k the abscissae of the points P_k for which

$$\varphi(x_k, P_k) = d - \varepsilon \text{ and } x_k < \beta, x_B < x_{k+1} < x_k$$

and by $\bar{x}_k$ the abscissae of the points Q_k for which

$$\varphi(\bar{x}_k, Q_k) = d - \varepsilon , \quad \bar{x}_k > \beta, \bar{x}_k < \bar{x}_{k+1} < x_C .$$

So we have two sequences of points $P_k = (x_k, d - \varepsilon)$ and $Q_k = (\bar{x}_k, d - \varepsilon)$, $P_k \to P$, and it is easily conceivable that $\lim_{x \to \beta - 0} \varphi(x, P) = d$. Hence according to Theorem A the sequence $\varphi(x, P_k)$ condenses on the integral $\varphi(x, P)$ and the coordinates of the point $P = (x_0, y_0)$ satisfy the conditions $x_B \leqslant x_0 < \beta$, $y_0 = d - \varepsilon$. Accordingly, $Q_k \to Q$ where the coordinates of the point Q satisfy the conditions:

$$\beta < x_Q \leqslant x_C, y = d - \varepsilon .$$

By Theorem A from [2], through the point Q it passes the associated integral with $\lambda(x)$, and the proof is complete.

3. Theorem 2. *If there exist two integrals of equation* (1.1), $y = \lambda(x)$ *defined for* $\alpha < x < \beta$, $y = \mu(x)$ *defined for* $\gamma < x < \delta$ *where* $\beta \leqslant \gamma$, *such that*

$$\lim_{x\to\beta-0} \lambda(x) = d, \quad \lim_{x\to\gamma+0} \mu(x) = d \left(\lim_{x\to\beta-0} \lambda(x) = c, \quad \lim_{x\to\gamma+0} \mu(x) = c\right)$$

then there exist lines $x = \eta$, $\beta \leqslant \eta \leqslant \gamma$ *and* $x = \varrho$, $\beta \leqslant \varrho \leqslant \gamma$ *which are the upper right-hand and the upper left-hand* (*the lower right-hand and the lower left-hand*) *asymptotes of furcation.*

Proof. We will prove the existence of the upper right-hand asymptote of furcation. One can prove the existence of the upper left hand asymptote of furcation similarly.

We denote by K the family of integrals defined for $x = \beta$ such that

$$\lim_{x\to\omega-0} \varphi(x) = d \text{ for a certain } \omega \in (\beta, \gamma]. \tag{3.1}$$

Two cases are possible: $K = \varnothing$, $K \neq \varnothing$.

If the case $K = \varnothing$ occurs then the line $x = \beta$ is the upper right-hand asymptote of furcation.

If case $K \neq \varnothing$ occurs then either a) there exists a point M lying on the line $x = \beta$ such that the integral passing through M does not belong to the family K, or b) through every point of line $x = \beta$ it passes an integral belonging to family K.

We will prove that if condition a) occurs then the asymptote of furcation exists. To this end we prove that there exists the integral $\varphi = \varphi(x)$ of the equation (1.1) such that the pair of functions φ and μ together with a certain straight line and some properly chosen points of the interval $(\beta, \gamma]$ satisfy the conditions given by the definition of the asymptote of furcation. Let $P_n = (\beta, y_n)$ be a sequence of points convergent to $P \in \Pi$, $P_n \to P$, $P_n \in \Pi$ and $P \in \Pi$.

If the integrals $\varphi(x, P_n)$ belong to the family K and $\varphi = \lim_{n\to\infty} \varphi(x, P_n)$ then

$$\lim_{x\to\psi-0} \varphi(x, P) = d \quad \text{and} \quad \psi \leqslant \gamma \tag{3.2}$$

or

$$\lim_{x\to\psi-0} \varphi(x, P) = c \quad \text{and} \quad \psi \leqslant \gamma. \tag{3.3}$$

Proof. If the sequence $\varphi(x, P_n)$ condenses at the point P then accordingly to Theorem A of [2] it condenses on the integral $\varphi(x, P)$. If $\psi > \gamma$ then for $n \geqslant N$ the integrals of the sequence $\varphi(x, P_n)$ are defined for $x > \gamma$ what is contradictory to condition (3.1). Assume that condition (3.3) occurs. There exists the line $y = d - \varepsilon$, $\varepsilon > 0$, ε so small that the straight line intersects the integrals $\lambda(x)$ and $\mu(x)$ but does not intersect the integral $\varphi(x, P)$ on the right side of the point P.

The integrals of the sequence $\varphi(x, P_n)$ passe through the line $y = d - \varepsilon$ at the points R_n (respectively) with coordinates $(x_n, d - \varepsilon)$ where $x_n < \gamma$. If we assume that $x_n < x_{n+1}$, then $R_n \to R$. This means that the sequence $\varphi(x, P_n)$ condenses on the integral $\overline{\varphi}(x, R)$ different from $\varphi(x, P)$. φ and $\overline{\varphi}$ are associated in the sense of Theorem A and definition of associated integrals. Therefore from Theorem 1 it follows the existence of an asymptote of furcation.

Consider the case (3.2). By our lemma, the set of points of the line $x = \beta$, through which pass the integrals of equation (1.1) belonging to family K, is connected.

We choose the sequence of points $\{D_n\}$ of that set that tends to the lower boundary D. Then

(3.4) $$\lim_{x \to \tau - 0} \varphi(x, D) = d \quad \text{and } \tau \leqslant \gamma$$

or

(3.5) $$\lim_{x \to \tau - 0} \varphi(x, D) = c \quad \text{and } \tau \leqslant \gamma \, .$$

The line $x = \tau$ is the upper right-hand asymptote of furcation in the case (3.4) or the lower right-hand asymptote of furcation in the case (3.5).

For (3.5) this has already been proved since the sequence $\{D_n\}$ is the special case of the sequence $\{P_n\}$.

We will prove that $x = \tau$ is the upper right-hand asymptote of furcation in the case (3.4). Assume that this is not true. Then there exists a point S on the line $x = \tau$ such that

$$\lim_{x \to \sigma - 0} \varphi(x, S) = d \quad \text{and} \quad \tau < \sigma \leqslant \gamma \, .$$

If the integral $\varphi(x, S)$ intersects the line $x = \beta$, then when the ordinate of the contact point $y_S > y_D$ we arrive at a contradiction with our lemma and if $y_S < y_D$ we arrive at a contradiction with the definition of the point D.

If the left-hand end ϱ of the saturated interval of the integral $\varphi(x, S)$ satisfies the inequality $\varrho > \beta$ then the integral $\varphi(x, D)$ passes through the line $x = \varrho$ at the point L with the coordinates $[\varrho, \varphi(\varrho, D)]$. On the line $x = \varrho$ we choose the sequence of points $L_n \to L$ with ordinates $y_n < \varphi(\varrho, D)$ and the integrals $\varphi(x, L_n)$ belong to the family K. The sequence $\varphi(x, L_n)$ condenses on the integral $\varphi(x, D)$ so that there exists an index N and the integral $\varphi(x, L_N)$ which is defined for $x = \beta$ and such that $\varphi(\beta, L_N) < \varphi(\beta, D)$ and this is contradictory to the definition of the point D.

Assume now that the case b) occurs.

Let us consider the integrals on the line $y = m$, $c < m < d$. Through the point $0_1(\beta, m)$ it passes the integral $\varphi(x, 0_1)$; this integral belongs to the family K.

We denote the right-hand end of the saturated interval by β_1 and by M_1 the difference $\beta_1 - \beta$. If through every point on the line $x = \beta_1$ it passes an integral belonging to the family K, we take the point $O_2(\beta_1, m)$ and the integral $\varphi(x, 0_2)$; by β_2 and $\beta_2 - \beta_1 = M_2 > 0$ we denote the right-hand end of the saturated interval of the integral $\varphi(x, 0_2)$. We prove that the sequence M_i has