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Functions with Separated Variables

Introduction

Certain properties of function with separated variables are discussed in the paper. The problem dealt with is closely related to the general problem, formulated by David Hilbert, concerning the representation of a function of n variables by means of functions depending on less than n variables. This problem was completely solved by A. Kolmogorov [1].

We are concerned with a particular case of the above problem when the function of n variables is to be represented as a sum of functions depending on at most one variable. While in [1] the functions are real-valued and depend on real variables, in our paper the ranges of functions are contained in an Abelian group G and their domains may be arbitrary sets. In sections 4 and 5 we restrict ourselves to the case when the domain is the n -dimensional Euclidean space R^n and G is the group of real numbers with addition (additively separable functions) or, with multiplication (multiplicatively separable functions). Section 1 contains definition of separability with respect to the pair of variables and with respect to the system of variables. The main result of the paper is obtained in section 2: we prove that a function which is separable with respect to every pair of variables is separable with regard to the system of variables. Section 3 deals with convergent sequences of separable functions. A differential criterion of separability is discussed in section 4 and section 5 is concerned with some problems of composite functions with separated variables.

Functions with separated variables play important role in the theory of partial differential equations (Bernoulli-Fourier method of separation of variables), in the probability theory (stochastic independence) and in the nomography.

Definitions, terminology

We denote by

$$(1) \quad f(x_1, \dots, x_n)$$

a function of n variables x_i ($i = 1, \dots, n$; $n \geq 1$) defined in the topological product

$$A = A_1 \times \dots \times A_n$$

of the non-void sets A_i . The values of the function (1) are supposed to belong to an Abelian group G . The group operation will be called addition and denoted by $+$ (we will also use the symbol Σ), the inverse operation will be called subtraction and denoted by $-$. The symbol 0 will stand for the neutral element.

Definition 1. For $n \geq 2$ we say that the function (1) is *separable in the set A with respect to the pair of variables x_j, x_k* , where

$$(2) \quad 1 \leq j \leq n, \quad 1 \leq k \leq n, \quad j \neq k,$$

if there are functions of $n-1$ variables

$$(3) \quad g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$$

$$(4) \quad h_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$$

defined respectively in the topological products

$$(5) \quad A_1 \times \dots \times A_{j-1} \times A_{j+1} \times \dots \times A_n$$

$$(6) \quad A_1 \times \dots \times A_{k-1} \times A_{k+1} \times \dots \times A_n$$

with values in the group G and satisfying in A the condition

$$(7) \quad f(x_1, \dots, x_n) = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) + h_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n).$$

Definition 2. We say that the function (1) is *separable in the set A with respect to the system of variables x_i* , if there are functions

$$(8) \quad p_i(x_i)$$

defined respectively in the sets A_i with values in the group G and satisfying the condition *

$$(9) \quad f(x_1, \dots, x_n) = \sum_{i=1}^n p_i(x_i).$$

Remark 1. It follows from the above definitions that the functions (3), (4) and (8) are not uniquely determined. We may, for instance, add to the function (3) an arbitrary function of $n-2$ variables

$$l_{jk}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$$

*) For $n = 1$, arbitrary function is separable.

defined in the topological product

$$A_{jk} = A_1 \times \dots \times A_{j-1} \times A_{j+1} \times \dots \times A_{k-1} \times A_{k+1} \times \dots \times A_n$$

and then subtract it from the function (4). We may, likewise, add to the functions (8) constants $c_i \in G$ satisfying the condition

$$(10) \quad \sum_{i=1}^n c_i = 0.$$

Remark 2. Let the functions $f_1(x_1, \dots, x_n)$ and $f_2(x_1, \dots, x_n)$ be separable in the sets Z_1 and Z_2 respectively according to the Definition 1 (2). If the set $Z = Z_1 \cap Z_2$ is not empty and the values of the functions belong to the same group G , then the linear combination with integer coefficients α and β is separable in Z according to the Definition 1 (2).

Now, we will prove two lemmas on the uniqueness of the functions (3) and (4) in the Definitions 1 and 2 (see Remark 1).

Lemma 1. *If for a function separable with respect to the pair of variables x_j, x_k the functions (3) and (4) as well as the functions*

$$g'_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), \quad h'_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$$

satisfy the conditions of Definition 1, then there exists a function of $n-2$ variables

$$(11) \quad x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{k-1}, x_{k+1}, \dots, x_n,$$

say

$$(12) \quad g_{jk}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{k-1}, x_{k+1}, \dots, x_n),$$

defined in the set A_{jk} such that

$$(13) \quad g'_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ + g_{jk}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{k-1}, x_{k+1}, \dots, x_n),$$

$$(14) \quad h'_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) = h_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \\ - g_{jk}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{k-1}, x_{k+1}, \dots, x_n).$$

Proof. It follows from the formula

$$f(x_1, \dots, x_n) \\ = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) + h_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \\ = g'_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) + h'_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$$

that

$$(15) \quad g'_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) - g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ = h_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) - h'_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n).$$

Hence we conclude that both sides of (15) are independent of x_j and x_k . Denoting the difference (15) by (12) we get (13) and (14).

Remark 3. It follows from Lemma 1 that functions (3) and (4) are uniquely determined by their values taken at $x_k = \dot{x}_k$ respectively at $x_j = \dot{x}_j$.

Lemma 2. If for a separable function with respect to the system of variables x_i the functions (8) as well as the functions $r_i(x_i)$ satisfy the conditions of Definition 2, then there exist constants $c_i \in G$ such that

$$(16) \quad \sum_{i=1}^n c_i = 0$$

and

$$(17) \quad r_i(x_i) = p_i(x_i) + c_i.$$

Proof. We choose an arbitrary point

$$(18) \quad P_0(\dot{x}_1, \dots, \dot{x}_n)$$

in the set A and we compute the value of the function (1) at that point. Then we get

$$(19) \quad f(\dot{x}_1, \dots, \dot{x}_n) = \sum_{j=1}^n p_j(\dot{x}_j) = \sum_{j=1}^n r_j(\dot{x}_j).$$

We fix now the index i and we compute the value of the function (1) at the point

$$(20) \quad Q_i(\dot{x}_1, \dots, \dot{x}_{i-1}, x_i, \dot{x}_{i+1}, \dots, \dot{x}_n).$$

It follows that

$$(21) \quad f(\dot{x}_1, \dots, \dot{x}_{i-1}, x_i, \dot{x}_{i+1}, \dots, \dot{x}_n) = \sum_{\substack{j=1 \\ j \neq i}}^n p_j(\dot{x}_j) + p_i(x_i) \\ = p_i(x_i) - p_i(\dot{x}_i) + \sum_{j=1}^n p_j(\dot{x}_j),$$

$$(22) \quad f(\dot{x}_1, \dots, \dot{x}_{i-1}, x_i, \dot{x}_{i+1}, \dots, \dot{x}_n) = r_i(x_i) - r_i(\dot{x}_i) + \sum_{j=1}^n r_j(\dot{x}_j).$$

Comparing formulas (19), (21), (22) we see that the formula (17) is satisfied by

$$(23) \quad c_i = r_i(\dot{x}_i) - p_i(\dot{x}_i).$$

From (19) and (23) we obtain (16).

Remark 4. It follows from Lemma 2 that for a separable function with respect to the system of variables x_i the function (8) are uniquely determined by their values taken at a chosen point.

We give now two examples specifying the set A and the group G .

a) A — the n -dimensional Euclidean space, G — the group of real or complex numbers with addition (function (1) is then called additively separable) or with multiplication (function (1) is then called multiplicatively separable).

b) A — n -fold topological product of the set of natural numbers, \mathcal{G} — the group of real or complex numbers with addition (additively separable n -fold sequence) or with multiplication (multiplicatively separable n -fold sequence).

Comparison of Definitions 1 and 2

A function which is separable in A with respect to the system of variables x_i according to Definition 2 is obviously separable with respect to every pair x_j, x_k . The following theorem shows that the inverse is also true.

Theorem 1. *If the function (1) is separable in A with respect to every pair x_j, x_k , then it is separable with respect to the system of variables x_1, \dots, x_n .*

Proof. We apply mathematical induction. By the hypothesis the theorem holds true when the number of variables is 2. Suppose it is true when the number of variables is $n-1$. It follows from the assumptions of theorem that the functions (3), (4) exist satisfying the condition (7). We choose in A an arbitrary point (18).

We first prove that for every $l = 1, \dots, n$ the function

$$(24) \quad f(x_1, \dots, x_{l-1}, \overset{\circ}{x}_l, x_{l+1}, \dots, x_n)$$

of $n-1$ variables is — according to Definition 2 — separable with respect to the system of variables

$$(25) \quad x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n$$

in the corresponding topological product. Let the index l be fixed and put in formula (7) $x_l = \overset{\circ}{x}_l$, where the indices j, k and l fulfil the condition

$$(26) \quad j \neq l, \quad k \neq l.$$

Then it follows that the function (24) is — according to Definition 1 — separable with regard to every pair x_j, x_k where j and k are arbitrary indices satisfying (2) and (26). Since (24) is a function of $n-1$ variables, we get from the induction assumption that it is separable with respect to the system of variables (25).

Putting $x_j = \overset{\circ}{x}_j$ in the formula (7) we get

$$(27) \quad f(x_1, \dots, x_{j-1}, \overset{\circ}{x}_j, x_{j+1}, \dots, x_n) = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ + h_k(x_1, \dots, x_{j-1}, \overset{\circ}{x}_j, x_{j+1}, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$$

Putting $x_k = \overset{\circ}{x}_k$ in the formula (7) we obtain

$$(28) \quad f(x_1, \dots, x_{k-1}, \overset{\circ}{x}_k, x_{k+1}, \dots, x_n) \\ = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{k-1}, \overset{\circ}{x}_k, x_{k+1}, \dots, x_n) \\ + h_k(x_1, \dots, x_{k-1}, \overset{\circ}{x}_k, x_{k+1}, \dots, x_n)$$

