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The Fundamental Solution for Second Order Elliptic Differential Equations

1. Introduction

Let D be a bounded domain in the Euclidean m-space R^m ($m \geqslant 3$). We shall denote its boundary by ∂D and its closure by $\bar{D}$. If Ω is a bounded set such that $\bar{\Omega} \subset D$, we shall write $\Omega \Subset D$. Let the partial differential operator L defined by

$$Lu = \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u$$

be elliptic in D. Denote by $\tilde{D}$ the set $\tilde{D} = (\bar{D} \times \bar{D}) \setminus (\partial D \times \partial D)$.

Definition 1 (cf. [5]). A function $G = G_D(x, y)$ will be called a *Green function* for the equation $Lu = 0$ with respect to the domain D if

1° $G_D(x, y)$ is continuous on $\tilde{D}$, for $x \neq y$,

2° $G_D(x, y) = 0$ for $x \in \partial D$, $y \in D$,

3° the integral

$$I_G(x) = \int_D G_D(x, y) f(y) dy$$

is of class C^2 on D, continuous on $\bar{D}$ and satisfies the identity

$$L[I_G(x)] = -f(x)$$

for all $f(x)$ in $C_0^{(0+\lambda)}(D)$ — class of Hölder continuous functions (with exponent λ) with compact support in D.

Definition 2. A domain D having the property that the Green function $G_D(x, y)$ there exists is termed a *regular domain*.

Let $D_0 \subset R^m$ be a bounded or unbounded domain. In particular, it may happen that $D_0 = R^m$. Suppose that there exists a sequence $D_1, D_2, \ldots$ of bounded and regular domains satisfying the following properties: $D_n \subset D_{n+1}$, $D_n \Subset D_0$, $D_n \to D_0$. Let $G_n(x, y)$ be the Green function for the equation $Lu = 0$ with respect to the domain D_n.

The main purpose of this paper is to show that the sequence $\{G_n(x, y)\}$ converges. The limit function will be called a *fundamental solution* for the equation $Lu = 0$ with respect to the domain D_0. Another purpose of our consideration is to obtain some estimation for the fundamental solution. This estimation is of particular interest if D_0 is unbounded, for example if $D_0 = R^m$.

2. Some properties of the Green function

Suppose that the coefficients of the operator L satisfy assumptions of Hop theorem (see [3]). Under the above conditions we have

Lemma 1 ([5]). *There is at most one Green function with respect to the domain D and the inequality $G_D(x, y) \geqslant 0$ holds for any two points x, y in D, $x \neq y$, provided $G_D(x, y)$ exists.*

Lemma 2 ([5]). *If $D \subset D_1$, then $G_D(x, y) \leqslant G_{D_1}(x, y)$ for $(x, y) \in \widetilde{D}$, $x \neq y$, provided $G_D(x, y)$ and $G_{D_1}(x, y)$ exist.*

Definition 3. We denote by $A(L, D_0)$ the family of all functions $h(x, y; a)$ continuous for $(x, y) \in \{(x, y): x \in D_0, y \in D_0, x \neq y\}$ which are twice continuously differentiable with respect to the first argument $x \in D_0$, $x \neq y$, and such that the following properties are satisfied:

1° $h(x, y; a) = 0(r^{-m+2+a})$ as $r \to 0$, where $r = |x-y|$, $0 < a < 1$,

2° $\frac{\partial}{\partial x_i} h(x, y; a) = 0(r^{-m+1+a})$, $\frac{\partial^2}{\partial x_i \partial x_j} h(x, y; a) = 0(r^{-m+a})$, as $r \to 0$,

3° $L_x h(x, y; a)$ is continuous with respect to $(x, y) \in D_0 \times D_0$, $x \neq y$ and $L_x h(x, y; a) < 0$ for $x, y \in D_0$, $x \neq y$,

4° $h(x, y; a) > 0$ for $x, y \in D_0$, $x \neq y$.

The symbol L_x denotes the operator L applied with respect to the variables x. This notation will be of importance later when we shall be dealing with functions of x and y, both in R^m. Under the hypotheses of theorem 3 of [5] the function $r^{-m+2+a} \in A(L, D_0)$. Let $\varrho_\varepsilon(t)$ be a $C^\infty[(0, \infty)]$ function such that

$$\varrho_\varepsilon(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant \frac{\varepsilon}{2}, \\ 0 & \text{for } t > \varepsilon. \end{cases}$$

Theorem 1. *Let D be a regular domain. Assume that the operator L is elliptic in $D_0 \supseteq D$, the coefficients of L are Hölder continuous on D_0 with exponent $\lambda (0 < \lambda < 1)$, $c(x) \leqslant 0$ on D_0 and there exists the function $h(x, y; a) \in A(L, D_0)$, where $a < \lambda$. Then for every point $y_0 \in D$ and for any number $\varepsilon > 0$ such that $K(y_0, 2\varepsilon) = \{y: |y-y_0| \leqslant 2\varepsilon\} \Subset D$, there exists a constant $M > 0$ such that*

$$G_D(x, y) \leqslant U_\varepsilon(x, y) + Mh(x, y; a)$$

for $x \in \bar{D}$, $y \in K(y_0, \varepsilon)$, *where*

$$U_\varepsilon(x, y) = (m-2)^{-1}\omega_m^{-1}[A(y)]^{-\frac{1}{2}}\Big[\sum_{i,j=1}^{m} A_{ij}(y)(x_i-y_i)(x_j-y_j)\Big]^{\frac{-m+2}{2}} \varrho_\varepsilon(|x-y|),$$

$[A_{ij}(y)]$ *is the inverse matrix to* $[a_{ij}(y)]$, ω_m *is the area of the unit sphere in* R^m *and* $A(y)$ *is the determinant of the matrix* $[A_{ij}(y)]$. *Constant* M *may be chosen independently of* D.

Proof. Let y_0 be a fixed point in D and ε a positive number such that $K(y_0, 2\varepsilon) \Subset D$. Consider the function

$$\Phi(x, y) = -U_\varepsilon(x, y) - Mh(x, y; a) + G_D(x, y).$$

The constant M will be defined later. Let $I_\Phi(x)$ denote the integral

(1) $$I_\Phi(x) = \int_D \Phi(x, y) f(y)\, dy,$$

where $f(y) \in C_0^{(0+\lambda)}(D)$. Then

$$I_\Phi(x) = -\int_D U_\varepsilon(x, y) f(y)\, dy - M\int_D h(x, y; a) f(y)\, dy + \int_D G_D(x, y) f(y)\, dy.$$

These integrals are in class $C^2(D)$ (see [3]). Hence

(2) $$L[I_\Phi(x)] = L\Big[-\int_D U_\varepsilon(x, y) f(y)\, dy\Big] - ML\Big[\int_D h(x, y; a) f(y)\, dy\Big] + L\Big[\int_D G_D(x, y) f(y)\, dy\Big].$$

$U_\varepsilon(x, y)$ is the Levi's function (see [3]). This together with the Poisson's theorem (see [3]) show that

(3) $$L\Big[-\int_D U_\varepsilon(x, y) f(y)\, dy\Big] = f(x) - \int_D L_x[U_\varepsilon(x, y)] f(y)\, dy.$$

Now we have

(4) $$L\Big[\int_D h(x, y; a) f(y)\, dy\Big] = \int_D L_x[h(x, y; a)] f(y)\, dy$$

(for details see [3]). From (2), (3), (4) and the definition of Green's function we obtain

$$L[I_\Phi(x)] = -\int_D L_x[U_\varepsilon(x, y)] f(y)\, dy - M\int_D L_x[h(x, y; a)] f(y)\, dy$$

or

$$L[I_\Phi(x)] = \int_D L_x[-U_\varepsilon(x, y) - Mh(x, y; a)] f(y)\, dy.$$

On the other hand,

$$L_x[U_\varepsilon(x, y)] = O(r^{-m+\lambda}) \quad \text{and} \quad L_x[h(x, y; a)] = O(r^{-m+\alpha});$$