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On a generalization of the convolution

Let $C[a, \infty)$ be the set of all real functions of a real variable on $[a, \infty)$, $K(x, t) \geqslant a$ a real continuous function defined in the domain D: $a \leqslant t \leqslant x < \infty$, and let $\varphi(x)$ be a real function of a real variable, of a bounded variation in every finite interval $a \leqslant x \leqslant X$. By $f * g$ we denote the function defined by the formula

$$(1) \qquad (f * g)(x) = \int_0^x f[K(x, t)]g(t)\,d\varphi(t)\,.$$

In the particular case, where $K(x, t) = x-t$, $\varphi(t) = t$ and $a = 0$, the function $f * g$ reduces to the ordinary convolution

$$(2) \qquad (f * g)(x) = \int_0^x f(x-t)g(t)\,dt$$

of the functions f and g.

As in well known, the set $C[0, \infty)$ with the operations of the addition of functions and the multiplication in the sense of formula (2) is an integral ring. The question arises, what additional conditions should be fulfilled by the functions K and φ in order that the set $C[a, \infty)$ with the operations of the addition and the multiplication in the sense of (1) be an integral ring. A partial answer to this question is given by the following

Theorem. Let, for every x, $K(x, t)$ be a strictly monotonic function of the variable t, and let $K(x, t)$ be continuous in the domain

$$D\colon\ a \leqslant t \leqslant x < \infty\,.$$

Further let $\varphi(x)$ be a monotonic and normalized [1] function in the interval $a \leqslant x < \infty$. If $C[a, \infty)$ has no zero divisors with respect to the operation (1) and if, moreover,

$$I * f = f * I \in C[a, \infty)$$

for every $f \in C[a, \infty)$, where $I(x) = 1$, then

[1] I.e. $\varphi(x) = \frac{1}{2}[\varphi(x+)+\varphi(x-)]$ for every x in $[a, \infty)$.

1° $\varphi(x)$ is a continuous and strictly monotonic function in the interval $a \leqslant x < \infty$;

2° the function K may be written as

$$K(x, t) = \varphi^{-1}(\varphi(x) - \varphi(t)),$$

where φ^{-1} is the inverse function to φ;

3° the set $C[a, \infty)$ is a commutative ring with respect to the operations of the addition and the multiplication in the sense of the generalized convolution (1).