

S. Gołąb

Functional equations in geometry

It is well known that the functional equations appear in many domains of mathematics.

In the sequel we shall understand the notion of functional equation in the sense which is expressly precised in the monograph of Aczél [1].

The purpose of my lecture is first to give a review of functional equations appearing in geometry, a review of course not pretending to be complete and excluding the theory of geometric objects in the exact meaning of this word. I exclude the functional equations of the theory of geometric objects with premeditation because themselves must occupy more than one hour.

The second purpose of my lecture is to quote some functional equations from my last geometric investigations. Some of them are till the present day not completely solved.

We meet functional equations in the foundations of geometry by treating it in an axiomatic way, although this fact is not accentuated expressly. In order to demonstrate this assertion I take the liberty to make use of a system of postulates for a flat euclidean geometry due to L. Dubikajtis [2]. He takes as primitive notions the notion of a point and the notion of the distance between two points.

If we shall denote the points by latin small letters and the distance between x, y by $\varrho(x, y)$, then one of the axioms of the system of Dubikajtis can be formulated in the following form.

If for a triplet of points x, y, z holds the equality

$$\varrho(x, y) + \varrho(y, z) = \varrho(x, z)$$

and if for the triplet (x, y, t) holds the relation

$$\varrho^2(x, y) + \varrho^2(y, t) = \varrho^2(x, t),$$

then for the triplet (y, z, t) holds the relation

$$\varrho^2(y, z) + \varrho^2(y, t) = \varrho^2(z, t).$$

It is easy to observe that all above relations represent functional equations and the unknown function φ is a scalar function of two variable points. Each of the aforementioned relations represents a functional equation with three independent variables. Thus we meet functional equations already in the foundations of geometry.

And now a short reconnaissance of historical nature. Somé supplementary details can be founded in the monograph of Aczél [1].

The functional equation

$$\varphi(x+y) = \varphi(x) \cdot \varphi(y)$$

of Cauchy type was used by Bolyai and Lobaczewski in the non-euclidean geometry. Lobaczewski also deduced in his geometry the formula for the so called angle of parallelity from the functional equation

$$\varphi^2(x) = \varphi(x+y) - \varphi(x-y).$$

With the Cauchy equation

$$\varphi(x+y) = \varphi(x) \cdot \varphi(y)$$

was employed Darboux in his investigations on analytic foundations of projective geometry.

Aczél first applied the functional equations for the definition of inner and exterior product of a couple of vectors in the euclidean space of three dimensions [3]. From the three following assumptions concerning the inner product $\varphi(x, y)$ and exterior product $f(x, y)$ of two vectors x, y Aczél deduces the classical form for φ and f up to the scalar coefficient. The assumptions are the following. We write them in the usual form, but the transposition of these properties into the functional equations containing the unknown functions φ and f makes no difficulties

$$\begin{array}{l} \text{I} \quad \left\{ \begin{array}{l} (x+y) \cdot z = x \cdot z + y \cdot z \\ (x+y) \wedge z = x \wedge z + y \wedge z \end{array} \right. \\ \text{II} \quad \left\{ \begin{array}{l} (\alpha x) \cdot y = \alpha(x \cdot y) = x \cdot (\alpha y) \\ (\alpha x) \wedge y = \alpha(x \wedge y) = x \wedge (\alpha y). \end{array} \right. \end{array}$$

In order to formulate the third hypothesis let us suppose that the group of motions in our space is a notion well known. If T is a motion, then Tx represents a vector uniquely determined. The third axiom admitted by Aczél can be formulated as follows

$$\text{III} \quad \left\{ \begin{array}{l} \varphi(x, y) = \varphi(Tx, Ty) \\ f(x, y) = f(Tx, Ty). \end{array} \right.$$

If the above axioms will be written in extenso we get a system of 6 functional equations for two unknown functions φ and f . In the main this problem reduces to the functional equation

$$\kappa(\xi + \eta) + \kappa(\xi - \eta) = 2\kappa(\xi) \cdot \cos \eta.$$

This equation was treated by Wilson, by the polish mathematician Kaczmarz and finally it was solved in a simple way and without any regularity supposition by Aczél [3].

Let us further remark that the so called Jensen equation

$$\varphi\left(\frac{x+y}{2}\right) = \frac{1}{2}[\varphi(x) + \varphi(y)]$$

has a geometric interpretation too.

The equation

$$\varphi(x+y) = \varphi(x)\varphi(y) - \sqrt{[1-\varphi^2(x)] \cdot [1-\varphi^2(y)]}$$

was object of investigation of Aczél and O. Varga in a problem of non-euclidean geometry. If we assume in the affine geometry that the inner product $\varphi(x, y)$ satisfies the axioms I_1, II_1 , then the distance $\varrho(p, q)$ of two points p, q can be determined if we set the following conditions [4]:

- 1) $p \neq q \Rightarrow \varrho(p, q) > 0$,
- 2) $\varrho(p, q)$ depends continuously on p, q ,
- 3) on each straight line $\varrho(p, q)$ is an additive function,
- 4) $\varrho(p, q) = \Phi(x(p), x(q))$ if $x(p)$ is the radius-vector of the point p and Φ is a function of a single variable.

It can be shown that under the above assumptions the function $\Phi(\xi)$ must have the following special form:

$$\begin{cases} \text{either } \Phi(\xi) = C \arccos \xi \\ \text{or } \Phi(\xi) = C \arccos h\xi. \end{cases}$$

It is known that Legendre deduced the formula for the area P of a rectangle with the sides x, y assuming the dependence

$$P = \Phi(x, y)$$

and that Φ is a positive and additive function. This leads of course to a system of functional equations and last of all to the Cauchy equation [5].

This problem can be stated more general assuming that in a vector-space we want to determine a functional $\Phi(x, y)$ representing the area of the triangle defined by the couple of vectors x, y . In the two-dimensional space the functional Φ can be defined as provided with the sign. In the spaces of higher dimension it is necessary to suppose $\Phi \geq 0$. One year ago I presented [6] at the meeting in Miskolc the result for $n = 2$ that, if we assume the homogeneity

$$\Phi(\lambda x, y) = \lambda \Phi(x, y)$$

the skew-symmetry

$$\Phi(y, x) = -\Phi(x, y)$$

and finally the relation

$$\Phi(x, y) = \Phi(x, -x - y)$$

having a very simple geometric interpretation, then we obtain without any regularity supposition the classical determinant formula for Φ . Mr Kucharszewski later weakened my hypotheses [7]. Although this result is equivalent to a former result of Carathéodory concerning the axiomatic characterization of determinants, it is deduced in a way much more simple and secondly it has an explicit geometric interpretation.

It is an interesting circumstance that the corresponding system of functional equations (again without any regularity suppositions) is more difficult to solve if we make the supposition $\Phi \geq 0$ and if we change suitably the other postulates.

Now I want to mention two geometric problems, different but leading to the similar integro-functional equations. The first of them issues to C. Popovici [8], the second is due to me. I had the opportunity to make a short report of it in Miskolc too.

C. Popovici put the question what meridian cuts have a rotational body having a constant density δ less than l in order that the depth of immersion in water (according to the law of Archimedes) be a given function of the total altitude of the body. More exactly speaking, if the body will be cut from above on the altitude x , then the depth of immersion shall be equal to $f(x) < x$, where f is given and φ is in quest of finding (see the figure).

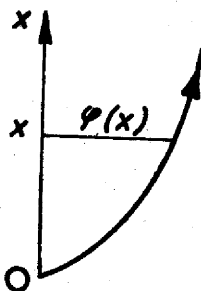


Fig. 1

This problem leads to the following equation of integro-functional type

$$\int_0^{f(x)} \varphi(y) dy - \int_0^x g(x, y) \varphi(y) dy = p(x),$$

where p and g are given non-negative functions having the physical meaning and $\varphi(x)$ is unknown. The author in his paper, written however without clearness, shows that the above equation has an infinite number of solutions which depend on an arbitrary function.

And now the second geometric problem quite different from the first leading to a similar but more complicated system of functional equations.

Let us take in the euclidean plane (x, z) a closed curve M having the origin $(0, 0)$ as centre of symmetry and the x -axis as axis of symmetry.

Let be the meridian of a surface S of rotation (z being the axis of rotation). Take any flat section of S going through the origin (determined by the angle α as marked on the figure) and denote its area by $P(\alpha)$. If the meridian is given by means of the equation

$$x = \varphi(z)$$

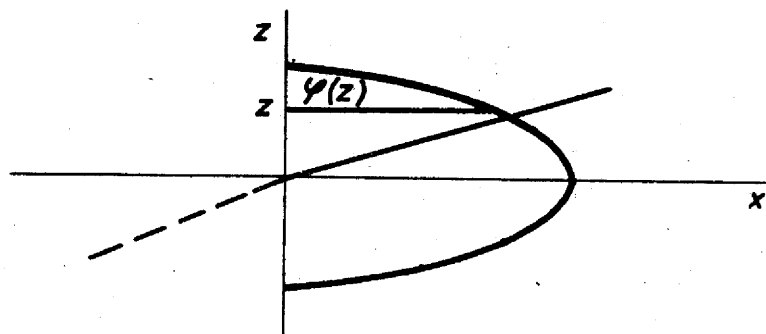


Fig. 2

then $P(\alpha)$ can be calculated. Now the question arises whether for a given function $P(\alpha)$ defined in the interval $\left[0, \frac{\pi}{2}\right]$ will exist a centric surface of rotation S such that its centric flat sections possess the given areas $P(\alpha)$. This problem leads to the following system of equations

$$\begin{cases} \int_0^{\varphi(z)} G[z, \Phi(z, y)] dy = F(z) \\ H\{y, z, \Phi(z, y), \varphi[\Phi(z, y)]\} = 0 \end{cases}$$

with two unknown functions $\varphi(z), \Phi(z, y)$. The functions $F(z), G(z, y)$ and $H\{y, z, u, v\}$ are known by virtue of geometric data of the problem.

In the above system the second equation is purely functional while the first is an integro-functional one. This system is difficult to solve. I don't know neither sufficient conditions for the existence of a solution nor sufficient conditions for the uniqueness of the solution. The geometric intuition is in this case a deceiving one.

Now I am going over to the question I was recently employed with namely to the problem of the definition of the inner product in vector-spaces by means of the norm of the vector. The converse question is trivial because having the inner product $\varphi(x, y)$ we put usually

$$\Omega(x) = \text{norm of } x \stackrel{\text{df}}{=} \sqrt{\varphi(x, x)}.$$

In the converse problem we have many possibilities to determine $\varphi(x, y)$ by help of $\Omega(x)$.

Usually we assume a definition referring to the well known formula of Carnot valid in euclidean spaces

$$(*) \quad \varphi(x, y) = \frac{1}{2} \{ \Omega^2(x) + \Omega^2(y) - \Omega^2(y-x) \} .$$

Doubless is this formula much restraining although it can lead to very interesting problems. As example I want to quote a theorem what was object of my communication on the Canadian Mathematical Society in Toronto in March of 1967.

Assume that the norm $\Omega(x)$ fulfills the following suppositions

$$\begin{cases} \Omega(x) \geq 0 \\ \Omega(x) = 0 \Leftrightarrow x = \theta \\ \Omega(\lambda \cdot x) = \lambda \Omega(x) \text{ for } \lambda > 0 \end{cases}$$

and assume moreover that our vector-space has a finite dimension.

If we define the inner product $\varphi(x, y)$ with the aid of formula (*) and if we suppose that the inner product verifies the distributivity condition

$$(1) \quad \varphi(x+y, z) = \varphi(x, z) + \varphi(y, z) ,$$

then it follows without any regularity supposition concerning $\Omega(x)$ that the Ω -metric must be a euclidean one.

Hence we can admit instead of (*) a supposition more general namely

$$(**) \quad \varphi(x, y) = \Phi[\Omega(x), \Omega(y), \Omega(y+x), \Omega(y-x)] ,$$

where Φ is a scalar function of four scalar variables and we can ask for the form of Φ if we impose to φ some properties. If these properties will be either the distributivity (1) or homogeneity

$$(2) \quad \begin{aligned} \varphi(\lambda \cdot x, y) &= \lambda \varphi(x, y) \\ \varphi(x, \lambda \cdot y) &= \lambda \varphi(x, y) \end{aligned}$$

or symmetry

$$(3) \quad \varphi(y, x) = \varphi(x, y)$$

then we get for Φ some functional equations. Its complete solutions is till now an open problem. It means that I dispose of local solutions while there are desirable solutions of global character.

Observe that if we don't suppose the symmetry

$$\Omega(-x) = \Omega(x)$$

and if we determine $\varphi(x, y)$ by help of (*), then the inner product φ will be in general non-symmetric and will not have the homogeneity property (2).

The general definition of φ by help of (**) can be called a definition of algebraic character.

We can imagine a definition having a different character of analytic structure. This other definition is suggested by some facts known from the Minkowski geometry.

We can namely assume

$$(***) \quad \varphi(x, y) = \Psi\{x, y, \text{grad}\Omega(x), \text{grad}\Omega(y)\}.$$

In this case the scalar function Ψ depends on $4 \cdot n$ variables if n denotes the dimensions of our vector-space. Of course the above definition supposes on the contrary to the definition (***) some regularity of the Ω -metric, namely the existence of the $\text{grad } \Omega(x)$.

Observe that the assumption of a concrete form of Φ and Ψ and the question when both definitions (***) and (***) are equivalent leads for the unknown function Ω to a differential-functional equation which is very difficult, till now not solved.

Finally some words about equations met recently investigating an elementary problem arising in the axiomatic foundation of affine geometry. In the simplest case we have there to do with a functional equation containing one unknown function of two variables

$$\varphi(x, y)$$

defined for all real values of x, y , globally reversible with respect to the second variable, it means that there exists a unique function

$$\varphi^{-1}(x, y)$$

satisfying the relations

$$y \equiv \varphi^{-1}[x, \varphi(x, y)] = \varphi[x, \varphi^{-1}(x, y)].$$

The requested equation is

$$\varphi^{-1}\{\varphi[(x, \varphi^{-1}(\varphi(x, z), y)), y]\} \equiv z.$$

The equality will hold for all real triplets (x, y, z) , I succeeded to solve this equation under the additional hypothesis that the unknown function φ is of regularity class C^1 , but I hope to success in weakening of regularity suppositions.

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