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Nr. CCXXIII PRACE MATEMATYCZNE. ZESZYT 14 1969

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Differentiable solutions of a linear functional equation in the indeterminate case

We shall consider the problem of differentiable solutions of the functional equations of the first order

$$\varphi[f(x)] = g(x)\varphi(x) + h(x)\,, \tag{1}$$

or

$$\varphi[f(x)] = g(x)\varphi(x)\,. \tag{2}$$

The functions f, g, h are assumed to be defined and continuous in an interval $I = \langle 0, a)$, $0 < a \leqslant \infty$, moreover, the function $f(x)$ is strictly increasing, $0 < f(x) < x$ in $(0, a)$ and the function $g(x)$ is positive in $(0, a)$.

We shall also assume that so called indeterminate case occurs, i.e. $|g(0)| = f'(0)$, since in the case $|g(0)| \neq f'(0)$ the problem of C^1-solutions has been solved. Namely, if $|g(0)| > f'(0)$ then equation (1) has the unique C^1-solution in I (cf. [1]), and if $|g(0)| < f'(0)$ then C^1-solution depends on an arbitrary function, i.e. values of a function $\varphi(x)$ may be arbitrarily prescribed in an interval $\langle f(x_0), x_0\rangle$, $x_0 \in I$, $x_0 \neq 0$, and then there exists the only extension of $\varphi(x)$ to a C^1-solution of (1) onto the whole interval I (cf. [2]). In the indeterminate case third possibility can also happen, viz. equation (1) can have a one-parameter family of solutions which are differentiable in I. The same can be said on continuous solutions in the interval I for which $|g(0)| = 1$ denotes the indeterminate case.

Thus let us assume that

$$g(0) = f'(0) = 1\,,$$

i.e. that we have the indeterminate case for both problems: continuity and differentiability of solutions. For this case I was looking for conditions of the existence of solutions of equation (1) (or (2)) which are either of class C^1 in I or differentiable in I or regular, i.e. continuous in I and having a derivative at the point zero. In the communication I shall only present the results concerning the existence of the only one-parameter family of regular solutions in the case when a continuous solution depends on an arbitrary function.

Assume the following expansion of the functions f, g, h:

$$\text{(3)} \qquad \begin{cases} f(x) = x - a_1 x^{k+1} + O(x^{k+1+\mu}) , \\ g(x) = 1 - a_1 x^k + O(x^{k+\nu}) , \qquad x \to 0+0 \\ h(x) = O(x^{k+1+\lambda}) \end{cases}$$

where $a_1, k, \mu, \nu, \lambda$ are positive constants, not necessarily integers.

Let $f^n(x)$ denote the n-th iterate of the function $f(x)$:

$$f^0(x) = x , \quad f^{n+1}(x) = f(f^n(x)) , \quad n = 0, 1, 2, \ldots, x \in I ,$$

and let us put

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)] .$$

Now we have the following

Theorem 1. Assume that all the conditions named above (especially, expansions (3)) hold. Then equation (1) has in I the only one-parameter family of regular solutions given by the formula

$$\text{(4)} \qquad \varphi(x) = -\sum_{n=0}^{\infty} h[f^n(x)]/G_{n+1}(x) + c \lim_{n\to\infty} f^n(x)/G_n(x)$$

(c is a parameter), whereas the continuous solution of equation (1) in I depends on an arbitrary function.

From this theorem follows a theorem on the regular solutions of equation (2) $(h(x) \equiv 0)$. But for equation (2) we have also the following

Theorem 2. Assume that the function $f(x)$ fulfils the hypotheses of the theorem 1 and so does the function $g(x)$, except the expansion (3) which will be replaced by the following one

$$g(x) = 1 - a_2 x^k + O(x^{k+\nu}) , \qquad x \to 0+0 ,$$

where $a_1 \neq a_2 > 0$. Then in the case where $a_1 < a_2$, equation (2) has I a regular solution depending on an arbitrary function, whereas in the case, where $a_1 > a_2$ the function $\varphi(x) \equiv 0$ is the only regular solution of equation (2) in I. However, a continuous solution of equation (2) always depends on an arbitrary function.

These theorems are proved in the paper [3]. In the proofs we make use of results of the first paper in which the indeterminate case is considered, that is of the joint paper by M. Kuczma and me [4]. We have dealt in this paper with continuous solutions of equation (1). By the use of an auxiliary equation we can reduce the problem of regular solutions to the problem of continuous ones, and it remains to investigate the convergence of the sequence and the series occuring in formula (4).

Finally, let us remark that similar theorems may be proved for solutions of (1) that are differentiable in I. For this purpose, generally speaking, one needs to assume on the place of (3) asymptotic expansions for the derivatives of the functions f, g, h (cf. [5]).

REFERENCES

[1] M. Kuczma, *A uniqueness theorem for a linear functional equation*, Glasnik Mat.Fiz. Astr. 16 (1961), 177-181.

[2] B. Choczewski, *Investigation of the existence and uniqueness of differentiable solutions of a functional equation.* Ann. Polon. Math. 15 (1964), 117-141.

[3] —, *Regular solutions of a linear functional equation in the indeterminate case,* Ann. Polon. Math. 21 (1969), 257-265.

[4] —, and M. Kuczma, *On the „indeterminate case" in the theory of a linear functional equation* Fund. Math. 58 (1966), 163-175.

[5] B. Choczewski, *On the differentiability of regular solutions of a linear functional equation.* Zeszyty Nauk. Uniw. Jag. Prace Mat. 13 (1969), 19-25.