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Functional equations in vector spaces

In this lecture we will summarize some of many results in the theory of functional equations in vector spaces. We divide this lecture in four parts.

I. ADDITIVE FUNCTIONS

The first, the most important and mostly explored is so called the Cauchy functional equation

$$f(x+y) = f(x)+f(y)\,. \tag{1}$$

A function which satisfies (1) will be called an additive function. Assuming that R is a set of all real numbers, $f\colon R \to R$ a continuous function, Cauchy [6] has obtained that $f(x) = xf(1)$.

If one assumes that f is bounded on an interval, then f is continuous (Darboux [7]). In [9] M. Fréchet has proved that measurability of f in the Lebesgue sense implies its continuity. Sierpiński [35] has proved an analogous theorem for convex functions. A. Ostrowski [34] has proved that boundedness of f on a set of positive measure implies its continuity.

S. Kurepa [18] has observed that boundedness of f on a set A implies that f is bounded on $A+A = \{a+b\colon\ a, b \in A\}$. Hence if $A+A$ contains an interval, what is the case if A has positive interior measure, then boundedness of f on such A implies its continuity. There are null sets A such that $A+A$ contains an interval.

P. Erdös observed that f may be bounded on a set A for which $A-A$ containes an interval and that f is not continuous. All solutions of (1) have been found by Hamel in terms of so called Hamel bases of R (R is a vector space over rational numbers and every algebraic basic set of R is called a Hamel basis of R).

I. Halperin [12] has asked whether (1) and $f\left(\frac{1}{x}\right) = \frac{1}{x^2}f(x)$ $(x \neq 0)$ imply continuity of f. The affirmative answer to this question was obtained independently by S. Kurepa [25] and W. Jurkat [17]. In [25] p. 30 two additive functions f, g are considered subjected to the condition: $g(t) = P(t)f(1/t)(t \neq 0)$ where P is a continuous function, $P(1) = 1$ and it was obtained that $f(t) + g(t) = 2tg(1)$ and $F(t) = f(t) - tf(1)$ is a derivative on R, i.e. that solutions are expressible in terms of derivatives on R. This result is generalized in [27]. An additive function h is a derivative on R if $h(xy) = xh(y) + yh(x)$ holds for all $x, y \in R$.

Additive functions from the point of our interest have been considered on abelian groups by J. Baker [4].

II. QUADRATIC FUNCTIONALS

Let R be the field of all real numbers, C the field of all complex numbers and $\varnothing$ any of them. Let X be a vector space over $\varnothing$.

A functional B: $X \times X \to \varnothing$ is called sesquilinear if

$$\begin{aligned} B(\lambda_1 x_1 + \lambda_2 x_2, y) &= \lambda_1 B(x_1, y) + \lambda_2 B(x_2, y) \\ B(x, \mu_1 y_1 + \mu_2 y_2) &= \bar{\mu}_1 B(x, y_1) + \bar{\mu}_2 B(x, y_2) \end{aligned} \tag{1}$$

holds for all $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \varnothing$ and all $x, y_1, x_1, y_1, x_2, y_2 \in X$. A sesquilinear functional B is hermitian functional if $B(x, y) = \overline{B(y, x)}$.

A functional $q: X \to \varnothing$ is termed quadratic if

$$q(x+y) + q(x-y) = 2q(x) + 2q(y) \qquad (x, y \in X). \tag{2}$$

If B is sesquilinear then $q(x) = B(x, x)$ is quadratic and

$$B(x, y) = \frac{1}{4}[q(x+y) - q(x-y)] \tag{3}$$

in the case $\varnothing = R$ and

$$B(x, y) = \frac{1}{4}[q(x+y) - q(x-y)] + \frac{i}{4}[q(x+iy) - q(x-iy)] \tag{4}$$

in the case $\varnothing = C$.

In any of these cases

$$q(\lambda x) = |\lambda|^2 q(x) \qquad (\lambda \in \varnothing;\ x \in X). \tag{5}$$

Relations (3) and (4) are well known in the case B is positive definite hermitian functional, i.e. $B(x, x) > 0$ for any $x \in X$, $x \neq 0$. It was M. Fréchet [10] who replaced the condition $B(x, y) = \overline{B(y, x)}$ in the definition of an inner product space by a metric relation. Simplification of his result was given by P. Jordan and J. v. Neumann [16]. The result is:

Let X be a complex vector space with distance defined in terms of a „norm" $|x|$, so that

$$|x+y| \leqslant |x| + |y|, \qquad |ix| = |x|, \ \lim_{t \to 0} |tx| = 0.$$

Then the identity

$$|x+y|^2+|x-y|^2 = 2\,|x|^2+2\,|y|^2$$

is characteristic for the existence of an inner product $B(x, y)$ connected with the norm $q(x) = |x|^2$ by the relation (4).

This result is usually quoted as necessary and sufficient condition for a normed space to be an inner product space.

Israel Halperin [12] in the New Scottish Book has raised the following question:

Assume that $q\colon X \to \emptyset$ satisfies (2) and (5). Define B by (3) in the case $\emptyset = R$ and by (4) in the case $\emptyset = C$. Is then B sesquilinear functional with the property that $B(x, x) = q(x)$?

In [25] in was proved that the answer to the above question is affirmative in the case $\emptyset = C$ and negative in the case $\emptyset = R$. The similar problem was considered also by A. Gleason [11].

If $R = \emptyset$ and $\{e_\alpha\colon 1 \leqslant \alpha < \Omega\}$ is an algebraic basic set in X, then

$$q\Big(\sum t_\alpha e_\alpha\Big) = \sum_{1\leqslant\alpha,\beta<\Omega} b_{\alpha\beta} t_t t_\beta + \sum_{1\leqslant\alpha<\beta<\Omega} \begin{vmatrix} a_{\alpha\beta}(t_\alpha), & a_{\alpha\beta}(t_\beta) \\ t_\alpha \quad , & t_\beta \end{vmatrix} \tag{6}$$

holds for all $t_\alpha \in R$, where the sum is actually finite, $b_{\alpha\beta} = b_{\beta\alpha}$ are real numbers and $a_{\alpha\beta}\colon R \to R$ are derivatives.

If in addition $\sup |q(x)| < \infty$ $(x \in \Delta)$ for every segment $\Delta \subset X$, then in (6) the second sum equals to zero and q is obtainable through a sesquilinear functional. Conversely by having derivatives $a_{\alpha\beta}$ and numbers $b_{\alpha\beta}$ by (6) is defined a quadratic functional for which (5) holds true [25].

By use of the above quoted theorem J. Baker [3] has found all quadratic functionals for which (5) is replaced by $|q(\lambda x)| = |\lambda^2 q(x)|$ and even for which the continuity of $\lambda \to q(\lambda x)$ is assumed only.

The equation (2) has been solved by J. Aczél [2] without any further assumptions in the case $X = R$.

III. EXPONENTIAL FUNCTIONS

A. Cauchy [6] has considered a function $f\colon R \to R$ such that $f(x+y) = f(x)f(y)$, $f(0) = 1$. If f is continuous then $f(x) = \exp Ax$ with a constant A. The same result is obtained if f is measurable in the Lebesgue sense.

If f is continuous on a bounded and closed set T then f is continuous on $T+T$ [18]. If $R \ni t \to U(t)$ is weakly continuous representation of the additive group R by unitary operators in a Hilbert space X, then $U(t) = \exp t(iA)$ with a selfadjoint operator A (M. H. Stone's theorem [37]). J. v. Neumann [33] has proved that the weak continuity in the Stone theorem can be replaced by the weak measurability in the Lebesgue sense provided that X is separable.

B. Sz. Nagy [38] considered a representation $R \ni t \to A(t)$, where $A(t)$ is a bounded selfadjoint operator. Assuming that $t \to A(t)x$ is measurable on a segment Δ for each $x \in X$ he obtained $A(t) = \exp tA$ with a bounded selfadjoint operator A.

The essential step was made by E. Hille [14] in 1938, who considered a semigroup

$$A(t+s) = A(t)A(s) \qquad (t, s \geqslant 0)$$

where $A(t)$ is a bounded selfadjoint operator. If $A(t)$ is continuous, then $A(t) = \exp tA$ $(t \geqslant 0)$ with an unbounded $A = A^*$.

S. Kurepa in [19] considered a semigroup $t \to A(t)$ of unbounded selfadjoint operators with a dense invariant set for all $A(t)$. Assuming that $t \to (A(t)x, x)$ is bounded on a set T, and that $T+T$ contains an interval it was obtained that $A(t) = \exp tA$.

Exponential functions are related to the differential equation

$$x' = Ax\,, \qquad x(0) = x_0\,. \tag{7}$$

The solution is $x(t) = e^{tA} x_0$ provided that the „infinitesimal generator" A is a number, or a constant $n \times n$ matrix or a bounded operator on a Banach space.

The general result in that respect is covered by the Hille-Yosida-Phillips theorem [15]:

A necessary and sufficient condition that a closed linear operator A with dense domain in a Banach space X is the infinitesimal generator of a strongly continuous semi-group is that there exist real numbers M and ω such that for every $\lambda > \omega$, λ is in the resolvent set $\varrho(A)$ of A and

$$|(\lambda I - A)^n| \leqslant M(\lambda - \omega)^{-n} \qquad (n = 1, 2, \ldots)\,. \tag{8}$$

There are extensions of the semigroup theory to locally convex vector spaces (see Yosida [44]). We quote here a result due to F. Vajzović ([43], [42]).

Let X be an inverse limit of Banach spaces $X_1, X_2, \ldots$ and $|x|_1 \leqslant |x|_2 \leqslant \ldots$ $(x \in X)$. Assume that $\{T(t)\colon t > 0\}$ is a family of continuous operators from X into X such that

a) $T(t+s) = T(t)T(s)$ $(t, s > 0)$

b) $t \to T(t)x$ is strongly measurable on $[a, b]$.

Then, for every integer $k > 0$, there is $n \geqq n(a, b)$ such that

$$|T(t)x|_k \leqslant n|x|_n \qquad (x \in X, t \in [a, b])\,.$$

Furthermore $t \to T(t)x$ is strongly continuous for each $x \in X$.

If in addition for each $k = 1, 2, \ldots$ and each $x \in X$, there are $M_k = M_k(x)$, $a_k = a_k(x)$ such that

$$|T(t)x|_k \leqslant M_k e^{a_k t} \qquad (t > 0)$$

then for each $x \in X$

$$|e^{t\frac{T(s)-1}{s}} x - T(t)x|_k \to 0 \text{ as } s \to 0 \qquad (k = 1, 2, \ldots).$$

A functional equation

$$T_1(t+s)\,T_2(t-s) = T_3(t)\,T_4(s)$$

has been treated in [28] under the assumption that $T_i(t)$ is an element of a Banach algebra. Sėe also [40].

IV. COSINE FUNCTIONAL EQUATION

This is an equation of the form

$$f(x+y)+f(x-y) = 2f(x)f(y)\,, \qquad f(0) = 1\,. \tag{9}$$

If $f\colon R \to R$ is continuous then $f(x) = \cos Ax$ with A as a real or pure imaginary number (A. Cauchy [6]). Assuming that f is defined on the set $\{\frac{l}{2^k}\colon l, k = 0, \pm 1, \pm 2, \ldots\}$ and that $f(t)$ is a matrix of order $n \times n$, this equation is troughly studied in [21]. The generalization to a Hilbert space is done in [22]. Here we quote the following theorem:

Let X be a Hilbert space, $R \ni t \to N(t)$ a bounded normal operator on X and $R \ni t \to F(t)$ a bounded linear operator.

Assume:

a) 1 is an eigenvalue of $N(s)$ for all $s \in R$;

b) there is $t_0 \in R$ such that t_0 is not in the spectrum of $F(t_0)$;

c) $t \to F(t)$ is weakly continuous on R;

d) $F(t+s)+F(t-s) = 2F(t)N(s)$ $(t, s \in R)$.

Then $F(t) = A\cos tN + B\sin tN$, $N(t) = \cos tN$ where N is a normal operator and A, B are bounded operators [23].

G. Maltese [32] considered (9) under the assumption that f is defined on a locally compact Abelian group and that it is a bounded normal operator on a Hilbert space.

S. Kurepa [20] considered (9) under the assumption that $f\colon R \to \mathfrak{A}$ is measurable where $\mathfrak{A}$ is a Banach algebra. Under these conditions one obtains:

$$f(t) = 1 + \frac{at^2}{2!} + \frac{a^2t^4}{4!} + \ldots \tag{10}$$

As we know f is the cosine function provided that $b \in \mathfrak{A}$ exists such that $a = b^2$. In connection with this we have this theorem:

Let $\mathfrak{A}$ be a Banach algebra with an identity e. There is a Banach algebra $\mathfrak{A}'$ with an identity E and a mapping $\vartheta\colon \mathfrak{A} \to \mathfrak{A}'$ such that

1) $\vartheta(\alpha a + \beta b) = \alpha\vartheta(a) + \beta\vartheta(b)$

2) $\vartheta(ab) = \vartheta(a)\vartheta(b)$

3) $|\vartheta(a)| = |a|$

4) $\sigma[\vartheta(a)] = \sigma(a)$.

Furthermore for any $a \in \mathfrak{A}$ and any natural number n, there is $T \in \mathfrak{A}'$ such that

I $T^n = \vartheta(a)$

II $ba = ab \Rightarrow \vartheta(b)T = T\vartheta(b)$.

This theorem enables one to clame that (10) is a cosine function in $\mathfrak{A}'$ [24].

In [43], [42] F. Vajzović has proved the following result:

Let X be a Fréchet space, $T(t)(t \in R)$ a continuous linear operator from X into X. Assume that:

1. $T(t+s)+T(t-s) = 2T(t)T(s)$, $T(0) = 1$ $(t, s \in R)$.
2. $T(t)x$ is strongly measurable on R for all $x \in X$. Then

I for any $a > 0$ and any natural number k, there is $n = n(a, k)$ such that

$$|T(t)x|_k \leqslant n\,|x|_n\,, \qquad x \in X\,, \qquad t \in \left[-\frac{a}{2}, \frac{a}{2}\right].$$

In the case X is a Banach space $|T(t)| \leqslant N \exp \alpha t$ with some constants α and N.

II. $t \to T(t)x$ is strongly continuous on R for each $x \in X$.

III. There exists a closed operator A with a dense domain $D(A)$ in X such that

$$T(t)x - x = \int_0^t (t-s)\,T(s)\,Ax\,ds\,, \qquad \big(x \in D(A)\big)\,. \tag{11}$$

If X is a Banach space, then

$$R_\lambda(\lambda^2 I - A)x = x \qquad \big(x \in D(A)\big)$$

$$(\lambda^2 I - A)R_\lambda x = x \qquad (x \in X)$$

$$\lambda^2 R_\lambda x \to x \quad \text{as} \quad \lambda \to \infty$$

where for sufficiently large λ

$$R_\lambda = \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\,T(t)\,x\,dt\,.$$

M. Sova [36] considered the equation (9) under the assumption that $x \geqslant y$ and $x, y \geqslant 0$. He proved the analogous theorem to the Hille-Yosida-Phillips theorem. Here is his main result:

Let X be a Banach space, M, ω two non negative numbers and A a linear operator defined on $D(A) \subseteq X$ into X. If

(α) for each $\mu > \omega^2$, $\mu I - A$ is regular,

(β) there is a dense set $D \subseteq X$ such that for each $x \in D$,

$$\mu(\mu I - A)^{-1}x \to x \text{ as } \mu \to \infty \;\; (\mu > \omega^2)\,,$$

(γ) for every $\lambda > \omega$ and every $n = 0, 1, 2, \ldots$

$$\left|\frac{d^n}{d\lambda^n}\left[\lambda(\lambda I - A)^{-1}\right]\right| \leqslant \frac{Mn!}{2}\left[\frac{1}{(\lambda+\omega)^{n+1}} + \frac{1}{(\lambda-\omega)^{n+1}}\right],$$

then, there is a cosine function $t \to F(t)$ such that

1. $|F(t)| \leqslant M ch\, \omega t \quad (t \geqslant 0)$,

2. $2 \lim\limits_{t \downarrow 0} \dfrac{F(t)x - x}{t^2} = Ax \quad (x \in D)$.

There are interesting applications of these results to differential equations.

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