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**On some applications of the theory of distributions  
in functional equations**

The equation

$$(1) \quad f(u + \varphi(t), v + \psi(t)) + f(u + \varphi(t), v - \psi(t)) + f(u - \varphi(t), v + \psi(t)) + \\ + f(u - \varphi(t), v - \psi(t)) - 4f(u, v) = 0$$

is a natural generalization of the Haruki equation

$$f(u+t, v+t) + f(u+t, v-t) + f(u-t, v+t) + f(u-t, v-t) - 4f(u, v) = 0$$

(Not only their form but also their geometrical interpretation is similar).

The most general continuous solution of Haruki's equation is

$$(2) \quad f(u, v) = auv(u^2 - v^2) + bu(3v^2 - u^2) + cv(3u^2 - v^2) + d(u^2 - v^2) + \\ + ev + fu + gv + h,$$

where  $a, b, c, d, e, f, g, h$  are arbitrary constants (cf. [1]).

It can easily be shown that in the case  $\varphi(t) = at$ ,  $\psi(t) = \beta t$  ( $a \neq 0, \beta \neq 0$ ) all the solutions of equation (1) can be obtained from those of the Haruki equation by the linear mapping  $u' = au$ ,  $v' = \beta v$ . However, the most general continuous solution of equation (1) has usually the form

$$(3) \quad f(u, v) = Auv + Bu + Cv + D,$$

where  $A, B, C, D$  are arbitrary constants.

**Theorem 1.** If  $\varphi(a) = \psi(a) = 0$ ,  $\varphi'(a)\psi'(a) \neq 0$ ,  $\varphi'(a)\psi''(a) \neq \varphi''(a)\psi'(a)$ , and if  $\varphi(t), \psi(t) \in C^3$  in an interval  $\Delta$  such that  $a \in \text{int}\Delta$ , then the most general continuous solution of equation (1) has form (3).

**Theorem 2.** If  $\varphi(a) = \psi(a) = 0$ ,  $\psi'(a) = 0$  (or  $\varphi'(a) = 0$ ),  $\varphi'(a)\psi''(a) \neq 0$  (or  $\varphi''(a)\psi'(a) \neq 0$ ),  $4\varphi'(a)\varphi'''(a) + 3\varphi''(a)^2 \geq 0$  (or  $4\psi'(a)\psi'''(a) + 3\psi''(a)^2 \geq 0$ ), and if  $\varphi(t), \psi(t) \in C^4$  in an interval  $\Delta$  such that  $a \in \text{int}\Delta$ , then the most general continuous solution of equation (1) has form (3).

Both these theorems are examples of applications of some general theorems on the regularity of the distributional solutions  $f$  of the equations

$$\sum_{i=1}^k a_i(x, t) f(x + \varphi_i(t)) = b(x, t),$$

where  $x \in R^n$ ,  $t \in R^r$ ,  $n > 1$ ,  $r \geq 1$  (cf. H. Świątak [2], [3]).

Remark. If the assumptions of theorems 1 and 2 are satisfied, every locally integrable solution  $f$  of equation (1) is equal to a function (3) almost everywhere.

Problems.

1) Does the assumptions of theorems 1 and 2 guarantee that the most general locally integrable solution of equation (1) has form (3)?

2) What are the weakest assumptions which allow to prove that the most general continuous (or locally integrable) solution of equation (1) has form (3)? Is it  $\varphi(t) \neq at$ ,  $\psi(t) \neq \beta t$ ?

3) Does the assumption:  $\varphi(t) \neq at$ ,  $\psi(t) \neq \beta t$  allow to prove that the most general solution of equation (1) is

$$f(u, v) = g(u, v) + h(u) + k(v) + C,$$

where  $g(u, v)$  is an arbitrary function additive in both variables,  $h(u)$ , and  $k(v)$  are arbitrary additive functions, and  $C$  is an arbitrary constant?

#### REFERENCES

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