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**Local analytic solutions of the functional equation  
 $\varphi(z) = h(z, \varphi[f(z)])$  in multidimensional spaces**

The functional equation

$$(1) \quad \varphi(z) = h(z, \varphi[f(z)]),$$

with the unknown function  $\varphi: C^m \rightarrow C^m$ , ( $C$  is the space of  $n$  complex variables) is considered under the following hypotheses:

(I) The function  $f: C^m \rightarrow C^m$  is analytic in a neighbourhood of  $z = 0$  and  $f(0) = 0$ .

(II)  $|s_i| < 1$ ,  $i = 1, \dots, n$ , where  $s_i$  are the characteristic roots of the matrix  $f'(0) = \left[ \frac{\partial f_i}{\partial z_j}(0) \right]_{i,j \leq n}$ .

(III) The function  $h: C^m \times C^m \rightarrow C^m$  is analytic in a neighbourhood of  $(z, w) = (0, 0)$  and  $h(0, 0) = 0$ .

(IV) There exists a formal solution

$$(2) \quad \varphi(z) = [\varphi^k(z_1, \dots, z_n)]_{k \leq m} = \left[ \sum_{\mu=1}^{\infty} \sum_{j_1, \dots, j_{\mu}=1}^n c_{j_1 \dots j_{\mu}}^k z_{j_1} \dots z_{j_{\mu}} \right]_{k \leq m}$$

of equation (1).

We put  $s_0 = \max |s_i|$  and we choose a positive integer  $p$  in such a manner that

$$\left( \sum_{k,l=1}^m \left| \frac{\partial h^k}{\partial w^l}(0, 0) \right|^2 \right)^{1/2} s_0^p < 1.$$

**Theorem 1.** If hypotheses (I)-(IV) are fulfilled, then for every system  $[c_{i_1 \dots i_q}^k]_{k \leq m, i_1 \dots i_q \leq n}$ ,  $q = 1, \dots, p$ , of coefficients in the formal solution (2) of the equation (1), there exists exactly one solution  $\varphi(z)$  of equation (1) defined and analytic in a neighbourhood of  $z = 0$  and such that

$$\varphi(0) = 0, \quad \left[ \frac{\partial^q \varphi^k}{\partial z_{i_1} \dots \partial z_{i_q}}(0) \right]_{k \leq m, i_1, \dots, i_q \leq n} = [c_{i_1 \dots i_q}^k]_{k \leq m, i_1, \dots, i_q \leq n}, \quad q = 1, \dots, p.$$

Now we specify the function  $f(z)$  to  $f(z) = Sz$ , where  $S$  is the matrix

$$S = \begin{bmatrix} S_1 & 0 & \dots & 0 \\ 0 & S_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & S_n \end{bmatrix},$$

and we assume that

(II')  $|s_i| \leq 1$ ,  $i = 1, \dots, n$ .

(IV')  $|\lambda_k| < 1$ ,  $k = 1, \dots, m$ , where  $\lambda_k$  are the characteristic roots of the matrix  $\left[ \frac{\partial h^k}{\partial w^i}(0, 0) \right]_{k, l \leq m}$ .

Condition (II') is weaker than the corresponding hypothesis (II).

Theorem 2. Let hypotheses (II'), (III) and (IV') be fulfilled. Then there exists exactly one solution  $\varphi(z)$  of equation

$$\varphi(z) = h(z, \varphi(Sz))$$

analytic in a neighbourhood of  $z = 0$  and such that  $\varphi(0) = 0$ .

Now we return to the case of a general  $f(z)$ . As previously,  $s_i$  will denote the characteristic roots of the matrix  $f'(0)$ .

We assume that

(V) There exists an analytic solution  $\gamma: C^m \rightarrow C^m$  in a neighbourhood of  $z = 0$  of the equation

$$\gamma[f(z)] = S\gamma(z)$$

such that  $\gamma(0) = 0$  and  $\gamma'(0)$  is a non-singular matrix.

Theorem 3. Suppose that hypotheses (I), (II'), (III), (IV') and (V) are fulfilled. Then there exists exactly one analytic solution  $\varphi(z)$  of equation (1) such that  $\varphi(0) = 0$ .

These results generalize our earlier one concerning the case  $n = m = 1$  which has appeared in *Annales Polonici Mathematici*, Volume 19. That result in turn, was a generalization of the classical result of Koenigs from the year 1884, concerning the existence and uniqueness of local analytic solutions of Schröder's functional equation.