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On meromorphic solutions of a functional equation

We consider the problem of the existence and uniqueness of the meromorphic solutions in the domain U for the equation

$$\varphi[f(z)]-g(z)\varphi(z)=h(z) \tag{1}$$

where $\varphi(z)$ is the unknown function and $f(z)$, $g(z)$, $h(z)$ are known functions of one complex variable z.

We assume:

(I) the function $f(z)$ is analytic in the bounded domain U such that $f(U)\subset U$ and there exists exactly one fixpoint $a\in U$ of the function $f(z)$. Moreover, we suppose that $|f'(a)|<1$.

(II) $g(z)$ and $h(z)$ are meromorphic functions in the domain U.

This problem was investigated by Raclis [2] when $g(z)=-1$ and by Pranger [1] when $g(z)=\text{const.}$ under a little different assumptions concerning the domain U.

Theorem 1. Let hypotheses (I) and (II) be fulfilled. Let us suppose that the functions $g(z)$ and $h(z)$ are analytic at the point a and, moreover,

$$g(a)\neq 0,1,c^k \text{ for } k=1,2,3,\ldots, \quad \text{where} \quad c=f'(a).$$

Then there exists at least one meromorphic solution of eq. (1) in the domain U. In case $g(a)\neq c^k$ for $k=-1,-2,\ldots$ there exists exactly one meromorphic solution.

Theorem 2. We assume (I) and (II). Let us suppose that $g(z)$ is analytic function at the point a, and $h(z)$ has a pole of order q at the point a. Moreover, let $g(a)\neq 0, c^k$ where $k=-q,-q+1,-q+2,\ldots$.

Then in case $c\neq 0$ there exists exactly one meromorphic solution of eq. (1) and the point a is its pole of the order q. In case $c=0$ there may be no solutions of eq. (1).

Theorem 3. We suppose (I) and (II) and, moreover, let $g(z)$ have a pole at the point a and let $h(z)$ be analytic at the point a.

Then there exists a meromorphic solution of eq. (1). In case $c\neq 0$ there exists the unique solution.

Theorem 4. Let (I) and (II) be fulfilled. Moreover, suppose that $g(z)$ has a pole of order p and $h(z)$ has a pole of order q at the point a.

Then

1. in case $p = q$ there exists a meromorphic solution of eq. (1). When $c \neq 0$ there exists exactly one solution and it is an analytic function at the point a.
2. in case $p > q$ when $c \neq 0$ there exists exactly one meromorphic solution of eq. (1). It has a pole of order $q-p$ at the point a.
3. in case $p < q$ there exists a meromorphic solution of eq. (1). When $c \neq 0$ then, there exists exactly one solution and it has a zero of order $p-q$ at the point a.

REFERENCES

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