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I. Stamate

Functional definitions for polynomials

The communication has dealt with the functional equation containing several unknown functions.

1. The functional equation

$$\varphi[a+(1-\lambda)\beta] = \frac{k}{\beta-a}\int_a^\beta f(t)\,dt \tag{1}$$

where λ and $k \neq 0$ are two constants, $f(z)$, $\varphi(z)$ being two real functions (summable or measurable) for any $a, \beta \in [a, b]$.

It has been shown that the multitude of the solutions of the equation is

1° $f(x) = \frac{c}{k}$, $\varphi(x) = c$;

2° $f(x) = \frac{1}{k}(ax+b)$, $\varphi(x) = ax+b$, $\lambda = \frac{1}{2}$.

The deduction of the solutions, in all cases, is performed, using the functional equation introduced by T. Popoviciu (Mathematica, Cluj, t. 10 and 14), namely the equation

$$\sum_{i=1}^{n} a_i\varphi(x+a_i h) = 0 \tag{2}$$

if the figure s is in such a way that

$$\sum_{i=1}^{n} a_i = \sum_{i=0}^{n} a_i a_i = \ldots = \sum_{i=0}^{n} a_i a_i^{s-1} = 0 \text{ and } \sum_{i=0}^{n} a_i a_i^{s} \neq 0$$

and $\varphi(t)$ is a function of the above mentioned form, then the general solution of the functional equation (2) is the general polynomial of $s-1$ degree.

2. The functional equation

$$\frac{1}{\beta-a}\int_a^\beta f(t)\,dt = \tfrac{1}{2}\,[g(a_1)+h(\beta_1)]$$

where

$$\alpha_1 = \alpha + \lambda(\beta - \alpha)\,,\ \beta_1 = \beta - \lambda(\beta - \alpha)\,,\ 0 \leqslant \lambda \leqslant \tfrac{1}{2}$$

admits as multitude of solutions

1° $\lambda = \frac{1}{2}$, $f(x) = \frac{1}{2}(ax+b)$, $g(x) + h(x) = ax + b$;

2° $\lambda \neq \frac{1}{2}$ and $\lambda \neq \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)$, $f(x) = \frac{1}{4}ax^2 + \frac{1}{2}(2b + c)x$, $h(x) = ax + b$, $g(x) = ax + b + c$;

3° $\lambda = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)$, $h(x)$ and $g(x)$ being general polynomials of 3 degree and the $f(x)$ polynomial of 4 degree.

The work contains other functional equations of the above form, too, all of them having polynomials as multitude of solutions.