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Topological methods in the theory of functional equations in a single variable

The object of this lecture is a review of applications of fixed point theorems in proofs of existence and uniqueness theorems for functional equations in a single variable. These methods do not appear in the theory of functional equations so often as e.g. in the theory of differential equations. In J. Aczél's book *Lectures on functional equations and their applications* we did not find any mention on applications of fixed point theorems. I have also read a lecture on functional equations by A. D. Wallace from University of Florida. The author pays attention to lack of local existence theorems for functional equations. He writes, in particular, that he has found „no existence theorems for functional equations which are based upon fixed point theorems, contrary the case with differential equations, where, for example, one also uses fixed point theorems to prove the existence of periodic solutions. And, on the other hand, the problem of periodic solutions is often considered for functional equations".

Essentially, this may be regarded as one more characteristic feature of functional equations in several variables, that there are no problems to solving with the aid of fixed point theorems. In Aczél's and Wallace's works only functional equations in several variables are considered. For functional equations in a single variable, however, the situation becomes different. In the monographical book [1] on these equations, recently written by M. Kuczma, proofs of some existence theorems are based upon fixed point theorems.

So far as I know, the first paper concerning functional equations in which topological methods have been used is due to M. Bajraktarević [2]. By the use of the well known theorem of Banach-Cacciopoli he has proved the existence of the only bounded solution of a functional equation. A main part of further results concerning the object of our considerations was obtained by mathematicians working in Katowice and Kraków. The reason of it lies, undoubtedly, in the fact that in these centres the theory of functional in a single variable has been strongely developped during the recent ten years. A majority of results is due to M. Kuczma, who has created a systematic theory of functional equations in a single variable.

Let us write the following functional equation in a single variable

$$\varphi(x) = H(x, \varphi[f(x)]), \tag{1}$$

where x is a real variable. This is the equation of the first order. A family of solutions $\varphi(x)$ of equation (1) depends, first of all, on a set E we want the equation to be fulfilled, and on the function f. The set E has to fulfill the inclusion $f(E) \subset E$. If E does not contain fixed points of the function f, i.e. $f(x) \neq x$ for $x \in E$ then there exist infinitely many solutions of equation (1) which are continuous or of class C^r in E. Thus in this case no condition of regularity allows us to find a narrower family of solutions, in particular exactly one solution. But in the case $\xi \in E$, where $f(\xi) = \xi$, one can give conditions for the existence of the only solution in a given family of functions. Fixed point theorems find an application just in proofs of local existence and uniqueness theorems concerning solutions in a neighbourhood of a fixed point ξ of the function $f(x)$.

I am going to show, by two examples, an application of two principal fixed point theorems, that is the well known theorems of Banach-Cacciopoli and of Schauder.

Let us assume that

(A) The function $f(x)$ is defined in an interval I, $\xi \in I$ and that it belongs to class $S_\xi^r(I)$ of functions (cf. [1], p. 20), i.e. $f \in C^r(I)$ and it fulfils the inequalities $\xi < f(x) < x$ for $x > \xi$ and $x < f(x) < \xi$ for $x < \xi$; $x \in I$.

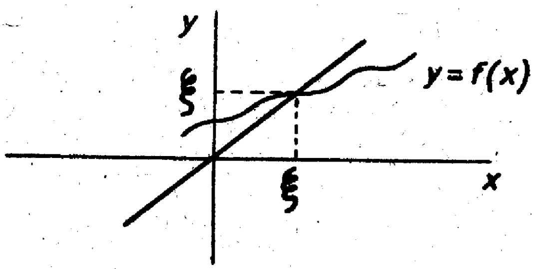

Fig. 1

Example I. Following M. Kuczma [3] we shall look for solutions of the linear functional equation

$$\varphi(x) = g(x)\varphi[f(x)] + \gamma(x) \tag{2}$$

which are of class C^r in I. In the paper [3] it was assumed that the values of $\varphi(x)$ and $\gamma(x)$ belong to a Banach vector space.

It is easy to see that the values $\eta_0, \ldots, \eta_r$ of the solution of equation (2) and its derivatives up to the order r at the fixed point ξ have to fulfil the conditions

$$\begin{cases} \eta_0 = g(\xi)\eta_0 + \gamma(\xi) \\ \eta_i = g(\xi)[f'(\xi)]^i \eta_i + \sum_{j=1}^{i-1} P_{ij}(\xi)\eta_j + \gamma^{(i)}(\xi)\,. \end{cases} \tag{3}$$

This condition may be obtained by the differentiation (i-times) of the equation (2) and then letting on $x = \xi$. Let us assume that the system (3) has a solution $\eta_0, \ldots, \eta_r$. We have the following

Theorem. Assume (A) and that the functions g and γ are of class C^r in I and the inequality

$$|g(\xi)[f'(\xi)]^r| < 1 \tag{4}$$

holds. Then there exists a neighbourhood of the point ξ, such that for every system $\eta_0, \ldots, \eta_r$ there exists exactly one solution $\varphi(x)$ of equation (2) being of class C^r in I_0 and fulfilling the conditions

$$\varphi(\xi) = \eta_0\,, \quad \varphi^{(i)}(\xi) = \eta_i\,, \quad i = 1, 2, \ldots, r\,. \tag{5}$$

Sketch of a proof. Consider the space R which elements are functions of class C^r in an interval I_0, whose length will be suitable chosen later, and which fulfil conditions (5). We introduce the metric

$$d(\varphi_1, \varphi_2) = \sup_{I_0} |\varphi_1^{(r)}(x) - \varphi_2^{(r)}(x)|$$

for $\varphi_1, \varphi_2 \in R$. The space R with this metric is complete, since the convergence in the space R is equivalent to the uniform convergence of functions and their derivatives up to the order r.

We now define the transformation

$$\Phi[\varphi](x) \overset{df}{=} g(x)\varphi[f(x)] + \gamma(x)$$

(the right-hand member of equation (2)). It maps R into itself (cf. (5)). We calculate the distance $d(\Phi[\varphi_1], \Phi(\varphi_2])$ using the formula of the r-th derivative. One can choose, owing to (4), the length of the interval I_0 in such a manner that this distance is less than $qd(\varphi_1, \varphi_2)$ where $q < 1$. On account of Banach theorem we obtain the unique C^r-solution of equation (2) as the limit of a sequence of succesive approximations.

If we want to obtain a similar result for equation (1) then we need to assume the inequality

$$\left|\frac{\partial H}{\partial z}(\xi, \eta_0)\right| [f^r(\xi)]^r < 1 \tag{6}$$

instead of (5), where $\eta_0 = H(\xi, \eta_0)$. We must also additionally assumed Lipschitz conditions for the derivatives $\partial^r H/\partial x^k \partial x^{r-k}$, $k = 0, \ldots, r$. I have proved the corresponding theorem in [4] using the similar method as it was described above.

If we omit the Lipschitz conditions, then we can only prove the existence of C^r-solutions by the use of the Schauder fixed point theorem. This is our

Example II. We take $r = 1$.

Theorem. Assume that $f(x)$ fulfils hypothesis (A) for $r = 1$, the function $H(x, z)$ is of class C^1 in a suitable set and assume (6) for $r=1$ and the inequality $f'(x) \leqslant 1$ in a neighbourhood of ξ. Then there exists a neighbourhood I_0