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Remarks on monotonic solution of a functional equation

The problem of the existence and uniqueness of monotonic solutions for the equation

$$\varphi[f(x)] = \varphi(x) + h(x) \tag{1}$$

was investigated by M. Kuczma, J. Burek and for $f(x) = x+1$ by F. John. M. Kuczma and J. Burek have proved that if the function f fulfils the hypothesis

(I) f is a real-valued function defined, continuous and strictly increasing in an interval (a, b), and such that

$$x < f(x) < b \quad \text{in} \quad (a, b),$$

and if the function h is defined, monotonic and has a constant sign in (a, b), then equation (1) has a monotonic solution.

The monotonicity of the function $h(x)$ is not a necessary condition of the existence of a monotonic solution of equation (1). It can be seen from the example (due to M. Kuczma) of the equation

$$\varphi(x+1) - \varphi(x) = \int_x^{x+1} e^{-u}(\sin u)^2 du$$

which has a monotonic solution $\varphi(x) = \int_a^x e^{-u}(\sin u)^2 du$, though the function $h(x)$ is not monotonic. Let us note that there does not exist a subdivision of $(-\infty, \infty)$ into a finite number of subintervals in which $h(x)$ is monotonic.

The situation changes if we assume that equation (1), with a fixed function h, has a monotonic solution for every function f from a larger class. Let us introduce the following hypotheses:

(II) h is a real-valued function defined in (a, b) and fulfilling the condition

$$\inf_{(a,b)} h(x) = 0 .$$

(III) F is a family of functions defined in (a, b) such that for every $x, y \in (a, b)$, $a < x < y < b$, there exists a function $f \in F$ fulfilling hypothesis (I) and the condition

$$f(x) = y .$$

An arbitrary iteration group of a function f fulfilling hypothesis (I) exemplifies the family satisfying hypothesis (III).

Now, we have the theorem:

Theorem 1. *Suppose that hypotheses (II) and (III) are fulfilled. Then equation* (1) *has an increasing solution for every function* $f \in F$ *if and only if one of the two possibilities occurs: either the function* h *is monotonic, or there exists a point* $x_0 \in (a, b)$ *such that* h *is decreasing in* (a, x_0) *(resp. in* $(x_0, b]$*) and increasing in* $[x_0, b)$ *(resp. in* $(a, x_0]$*).*

M. Kuczma has also proved that if the function $f(x)$ fulfils hypothesis (I) and the function $h(x)$ is monotonic and fulfils

$$\lim_{x \to b-} h(x) = 0 ,$$

then the monotonic solution of equation (1) is unique up to an additive constant. His result can be improved as follows.

Theorem 2. *Under hypotheses (I) and (II) every two increasing solutions of equation* (1) *in* (a, b) *differ by a constant.*