ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
Nr. CCXXIII PRACE MATEMATYCZNE, ZESZYT 14 1969

S. Czerwik

On sign preserving solutions of a functional equation

The object of the present report is the functional equation

(1) $$\varphi[f^n(x)]+A_1\varphi[f^{n-1}(x)]+\ldots+A_n\varphi(x) = F(x)\,,$$

where $\varphi(x)$ is the required function, $f(x)$ and $F(x)$ are known functions and A_i are constant real coefficients, $A_n \neq 0$. $f^k(x)$ denotes here the k-th iterate of the function $f(x)$.

We seek the solutions satisfying one of the conditions:

(2) $$\varphi(x) \geqslant 0\,,$$

(3) $$\varphi(x) \leqslant 0\,,$$

(4) $$\varphi(x) \geqslant G(x)\,,$$

(5) $$\varphi(x) \leqslant G(x)\,,$$

where $G(x)$ is a known function.

There may exist infinitely many such solutions of equation (1). In the present report we shall give some conditions of the uniqueness of solutions of equation (1) fulfilling one of the conditions (2)-(5).

Theorem 1. We suppose that the function $f(x)$ is defined in an interval I and satisfies the condition $f(I) \subset I$. If the characteristic polynomial of equation (1)

(6) $$W(\lambda) = \lambda^n+A_1\lambda^{n-1}+\ldots+A_n$$

has n real roots $\lambda_1, \ldots, \lambda_n$ such that

(7) $$\lambda_i < 0 \quad \text{for} \quad i = 1, \ldots, n\,,$$

and moreover

(8) $$\lim_{\nu\to\infty}\frac{F[f^\nu(x)]}{\lambda_0^\nu} = 0 \quad \text{for} \quad x \in I\,,$$

where $\lambda_0 \overset{df}{=} \min_i |\lambda_i|$, then equation (1) has at most one solution satisfying condition (2) resp. condition (3). The necessary condition for the existence of such a solution is the relation $F(x) \geqslant 0$ for $x \in I$ resp. $F(x) \leqslant 0$ for $x \in I$. If the solution actually does exist, then it is given by the formula

$$(9) \qquad \varphi(x) = (-1)^n \sum_{\nu_1=0}^{\infty} \dots \sum_{\nu_n=0}^{\infty} \frac{1}{\lambda_1^{\nu_1+1}} \dots \frac{1}{\lambda_n^{\nu_n+1}} F[f^{\nu_1+\dots+\nu_n}(x)] .$$

Theorem 2. Let the function $f(x)$ fulfil the hypotheses of Theorem 1. If characteristic polynomial (6) has n real roots λ_i satisfying condition (7) and moreover

$$(10) \qquad \lim_{\nu\to\infty} \frac{F^*[f^\nu(x)]}{\lambda_0^\nu} = 0 \quad \text{for} \quad x \in I ,$$

where $\lambda_0 = \min_i |\lambda_i|, F^*(x) = F(x) - \sum_{i=0}^{n} A_i G[f^{n-i}(x)] , \quad A_0 \mp 1 ,$

and $G(x)$ is a given function defined in I, then equation (1) may have at most solution satisfying condition (4) resp. (5). The necessary condition for the existence of such a solution is the relation $F^*(x) \geqslant 0$ in I, resp. $F^*(x) \leqslant 0$ in I. If the solution actually does exist, then it is given by the formula

$$(11) \qquad \varphi(x) = G(x) + (-1)^n \sum_{\nu_1=0}^{\infty} \dots \sum_{\nu_n=0}^{\infty} \frac{1}{\lambda_1^{\nu_1+1}} \dots \frac{1}{\lambda_n^{\nu_n+1}} F^*[f^{\nu_1+\dots+\nu_n}(x)] .$$

Theorem 3. Let the function $F(x)$ fulfil the condition

$$(12) \qquad \lim_{\nu\to\infty} F[f^\nu(x)] = 0 \quad \text{for every} \quad x \in I ,$$

and let

$$f(x) \neq x , \quad f(I) \subset I .$$

If moreover

$$(13) \qquad F(x) \geqslant 0 \text{ and } \Delta_{(f)} F[f^\nu(x)] \leqslant 0 \text{ for } x \in I \text{ and } \nu = 0, 1, \dots ,$$

resp.

$$(14) \qquad F(x) \leqslant 0 \text{ and } \Delta_{(f)} F[f^\nu(x)] \geqslant 0 \text{ for } x \in I \text{ and } \nu = 0, 1, \dots ,$$

where

$$\Delta_{(f)} F(x) = F[f(x)] - F(x) ,$$

then there exists exactly one solution $\varphi(x)$ of the equation

$$(15) \qquad \varphi[f^n(x)] + \dots + \varphi[f(x)] + \varphi(x) = F(x)$$

satisfying condition (2) resp. condition (3). This solution is given by the formula

$$\varphi(x) = -\sum_{\nu=0}^{\infty} \Delta_{\{f\}} F[f^{(\nu+1)}(x)]\,. \tag{16}$$

Theorem 4. Let the function

$$F^*(x) = F(x) - \sum_{i=0}^{n} G[f^i(x)]$$

fulfil conditions (12) and (13) resp. (14), and let $f(I) \subset I$, $f(x) \neq x$ in I. Then there exists exactly one solution $\varphi(x)$ of equation (15) satisfying condition (4) resp. (5). This solution is given by the formula

$$\varphi(x) = G(x) - \sum_{\nu=0}^{\infty} \Delta_{\{f\}} F^*[f^{(n+1)\nu}(x)]\,. \tag{17}$$