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On the stability of the linear functional equation

The problem of stability of a functional equation was first considered by D. Hyers [1], who has proved that the Cauchy functional equation is stable. We shall consider the stability of the linear functional equation

$$\varphi[f(x)] = g(x)\varphi(x) + F(x) \tag{1}$$

where f, g, F are given functions and φ is unknown function. We shall assume that the functions, f, g and F are continuous in an intervall I, $g(x) \neq 0$ in I, f is a strictly increasing function in I and there exists a point $\xi \in I$ such that

$$[f(x)-x](\xi - x) > 0 \quad \text{for} \quad x \in I,\ x \neq \xi$$

$$[f(x)-\xi](\xi - x) < 0 \quad \text{for} \quad x \in I,\ x \neq \xi$$

We say that equation (1) is stable in the sens of D. Hyers if there exists a positive number k such that for each positive number ε and each continuous ψ satisfying

$$|\psi[f(x)] - g(x) - F(x)| \leqslant \varepsilon, \qquad x \in I \tag{2}$$

there exists a solution φ of (1) such that we have

$$|\psi(x) - \varphi(x)| \leqslant k\varepsilon, \qquad x \in I. \tag{3}$$

The equation (1) is not stable in the above sens, because for given $\varepsilon > 0$ the function $\psi(x) = \varepsilon x$ satysfies (2), with $f(x) = x+1$, $g(x) = 1$, $F(x) = 0$ and $I = (-\infty, \infty)$, but it is not bounded, so that the inequality (3) can not be satisfied, for all solutions of (1) are bounded.

We shall modify the definition of stability as follows: we say that the equation (1) is stable if there exists a positive number k such that for each $\varepsilon > 0$ and each ψ continuous satisfying

$$\psi[f^n(x)] - G_n(x)\psi(x) - G_n(x) \sum_{i=0}^{n-1} \frac{F[f^i(x)]}{G_{i+1}(x)} \leqslant \varepsilon \tag{4}$$

where $G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)]$ there exists a continuous solution φ of (1) satysfying (3).

We shall consider three cases:

i) $G_n(x) \to G(x) \neq 0$ in I and G is continuous in I,

ii) There exists an intervall $J \subset I$ such that $G_n(x) \underset{J}{\rightrightarrows} 0$

iii) Neither i) nor ii) is satisfied.

Theorem I. In the case i) the equation (1) is stable if it has a continuous solution and if

$$\sup_{x \in I} \frac{1}{|G(x)|} < \infty . \tag{5}$$

Proof. If ψ satisfies (4), we have

$$|G_n(x)| \left| \frac{\varphi[f^n(x)]}{G_n(x)} - \psi(x) - \sum_{i=0}^{n} \frac{F[f^i(x)]}{G_{i+1}(x)} \right| \leqslant \varepsilon , \quad n = 1, 2, \ldots, x \in I .$$

As (1) has a continuous solution in I then the series $\varphi_0 = -\sum_{n=0}^{\infty} \frac{F[f^n(x)]}{G_{n+1}(x)}$ converges in I and φ_0 is a continuous solution of (1) in I. Then, if n tends to infinity, we obtain

$$\left| \frac{\psi(\xi)}{G(x)} - \psi(x) + \varphi_0(x) \right| \leqslant k\varepsilon , \quad x \in I \tag{6}$$

where $k = \sup_{x \in I} \frac{1}{|G(x)|}$. But $\frac{\psi(\xi)}{G(x)} + \varphi_0(x)$ is a continuous solution of (1), then (6) implies that $\varphi(x) = \frac{\psi(\xi)}{G(x)} + \varphi_0(x)$ satisfies (3).

Theorem 2. In the case ii) the equation (1) is stable in $\langle \xi, f(x_0) \rangle$ (or $\langle \xi, x_0 \rangle$) if it has a continuous solution and if there exists a point $x_0 \in I$ such that the interval $I_0 = \langle x_0, f(x_0) \rangle$ (or $I_0 = \langle f(x_0), x_0 \rangle$) is contained in J.

Proof. We may assume that ξ is the left end of I.i.e. $I_0 = \langle f(x_0), x_0 \rangle$. Suppose that ψ is a continuous solution of (4). Put

$$y_0 = \psi(x_0) . \tag{7}$$

Let φ_0 be a function defined and continuous on I_0, such that

$$\varphi_0(x_0) = y_0 \tag{8}$$

$$\varphi_0[f(x_0)] = g(x_0)\varphi(x_0) + F(x_0) . \tag{9}$$

We obtain from (4) and (9)

$$\begin{aligned} |\varphi_0[f(x_0)]| - \psi[f(x_0)]| &= |g(x_0)\varphi_0(x_0) + F(x_0) - \psi[f(x_0)]| \\ &= |g(x_0)\psi(x_0) + F(x_0) - \psi[f(x_0)]| \leqslant \varepsilon . \end{aligned} \tag{10}$$

For such a function φ_0 there exists exactly one continuous solution φ of (1) in I, such that

(11) $$\varphi(x) = \varphi_0(x) \quad \text{for} \quad x \in I_0 .$$

Let $x \in J \setminus \{\xi\}$. There exists a positive integer p, such that $x = f^p(t)$, where $t \in I_0$. Now, the inequality (3) can be proved by virtue of (1), (4), (10) and (11) as follows:

$$|\varphi(x) - \psi(x)| = |\psi[f^p(t)] - \varphi[f^p(t)]| \leqslant \Big|\psi[f^p(t)] - G_p(t)\psi(t) + \\ + G_p(t) \sum_{i=0}^{p} \frac{F[f^i(t)]}{G_{i+1}(t)}\Big| + |G_p(t)|\,|\psi(x) - \varphi(t)| \leqslant \varepsilon(1 + M)$$

where $M = \sup_p \sup_{t \in I_0} |G_p(t)|$. $M < \infty$ because G_n converges uniformly to zero in I_0. It is easy to see that the above inequality holds also for $x = \xi$.

In the case (iii) the equation (1) is not stable.

REFERENCES

[1] D. Hyers, *On the stability of the linear functional equation*. Prac. Mat. Acad. Sci. U.S. 27 (1941), 222-224.