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Nicoletti Boundary Value Problems for Systems of Linear Differential Equations with Distributional Perturbations

1. INTRODUCTION

In the present note we are concerned with the equation

$$x' = A(t)x + f, \tag{1}$$

where $A(t)$ is a matrix function and f is a vector distribution. It is well known that the initial value problem for differential equations with distributions as solutions does not have any sense. For the n-order linear differential equations this difficulty has been overcome by J. Kurzweil [1], who has replaced the initial value problem by a certain interpolation problem. In the present note we will consider the problem of the existence and uniqueness of solutions of (1) satisfying the so-called Nicoletti condition

$$\langle x, \varphi \rangle = r \tag{2}$$

(see Section 2). In the last part of this paper we state a theorem concerning the continuous dependence of solutions of (1), (2) on the right-hand side of (1) and the boundary condition (2).

2. NOTATIONS

Let (α, β) be an interval of real line R. By D_m we denote the linear space of all functions defined and continuous with the derivatives up to the order m in (α, β) and with compact supports. We say that $\{\psi_n\} \subset D_m$ *converges* to ψ, if there is an interval $[\alpha', \beta'] \subset (\alpha, \beta)$ such that $\operatorname{supp}\psi_n \subset [\alpha', \beta']$ $(n = 1, 2, \ldots)$ and if $\{\psi_n\}$ converges to ψ uniformly with derivatives up to the order m. Let $\widetilde{D}^l_m$ denote the linear space of all l-dimensional vector distributions of order m on (α, β). A sequence $\{x_n\} \subset \tilde{D}^l_m$ will be called *convergent* to x if $\langle x_n, \psi\rangle \to \langle x, \psi\rangle$ for every $\psi \in D_m$.

By D_{+m} we will denote the subset of D_m containing all functions ψ such that

$$\int_\alpha^\beta \psi(t)\,dt = 1\,, \quad \psi(t) \geqslant 0\,, \quad (\alpha < t < \beta)\,.$$

By the convergence in D_{+m} and in its l-order cartesian product D^l_{+m} we will always mean the convergence induced by the convergence in D_m. For a given $x = (x_1, \ldots, x_l) \in \tilde{D}^l_m$ and $\varphi = (\varphi_1, \ldots, \varphi_l) \in D^l_{+m}$ we set

$$\langle x, \varphi\rangle = (\langle x_1, \varphi_1\rangle, \ldots, \langle x_l, \varphi_l\rangle)$$

and

$$\Delta\varphi = [\inf(\bigcup_{i=1}^{l} \operatorname{supp}\varphi_i), \sup(\bigcup_{i=1}^{l} \operatorname{supp}\varphi_i)]\,.$$

By $C^{l\times l}_m$ we will denote the linear space of all $l\times l$ matrix functions $A(t) = \{a_{ij}(t)\}$ defined and continuous with the derivatives up to the order m in (α, β). A sequence $\{A_n\} \subset C^{l\times l}_m$ will be called *convergent* to A if it is almost uniformly convergent to A with all derivatives up to the order m in (α, β).

For an $l\times l$ real matrix $A(t) = \{a_{ij}(t)\}$, let

$$|A(t)| = \sqrt{\sum_{i,j=1}^{l} [a_{ij}(t)]^2}\,.$$

3. EXISTENCE AND UNIQENESS OF SOLUTIONS

Consider equation (1) and the boundary value problem (2).

Denote by $U(t) = \{u_{ij}(t)\}$ the fundamental matrix solution of the homogeneous system

(3) $$x' = A(t)x\,.$$

We have the following

Theorem 1. *Let* $A \in C^{l\times l}_m$ *and* $f \in \tilde{D}^l_{m+1}$. *For any function* $\varphi \in D^l_{+m}$ *such that*

(4) $$\int_{\Delta\varphi} |A(t)|\,dt < \frac{\pi}{2}$$

and for any $r \in R^l$, *there exists exactly one solution of problem* (1), (2).

Proof. The family of all solutions of equation (1) is given by the explicit formula

(5) $$x = U(t)C + U(t)g\,,$$

where $C \in R^l$ and g is a primitive of the distribution $U^{-1}(t)f$. By (5), for $\varphi \in D'_{+m}$ we have

$$(6) \qquad \langle x, \varphi\rangle = \langle U(t)C, \varphi\rangle + \langle U(t)g, \varphi\rangle .$$

Put

$$(7) \qquad d = \langle U(t)g, \varphi\rangle .$$

Now, by the mean-value theorem,

$$(8) \qquad \langle U(t)C, \varphi\rangle = \int_a^\beta \operatorname{diag}\varphi(t)\cdot U(t)C dt = H(\xi_1, \ldots, \xi_l)C ,$$

where

$$H(t_1, \ldots, t_l) = \begin{bmatrix} u_{11}(t_1) \ldots & u(t_{1l}) \\ \ldots \quad \ldots & \ldots \\ u_{l1}(t_l) \ldots & u_{ll}(t_l) \end{bmatrix}$$

for $t_i \in (\alpha, \beta)$ and $\xi_i \in \Delta\varphi$ $(i = 1, \ldots, l)$. The matrix

$$(9) \qquad Q = H(\xi_1, \ldots, \xi_l) = \int_a^\beta \operatorname{diag}\varphi(t)\cdot U(t)dt$$

is nonsingular. If this were not true, there would exist a vector $\Lambda = (\lambda_1, \ldots, \lambda_l) \in R^l$, $\Lambda \neq 0$, such that

$$(10) \qquad Q\Lambda = 0$$

and the function $z(t) = (z_1(t), \ldots, z_l(t)) = U(t)\Lambda$ would be a non-trivial solution of (3) satisfying the Nicoletti condition

$$z_i(\xi_i) = \sum_{j=1}^{l} u_{ij}(\xi_i)\lambda_j = 0 \quad (\xi_i \in \Delta\varphi;\ i = 1, \ldots, l).$$

This yields a contradiction, since by [2] it easily follows from (4) that the problem of Nicoletti has only the trivial solution. Equality (6), by (2), (7) and (8) takes the form

$$r = QC + d$$

or, equivalently,

$$C = Q^{-1}(r - d)$$

which completes the proof.

Remark 1. It is easy to see that only the nonsingularity of Q was important in this proof.

Remark 2. Theorem 1 remains true if A is a locally summable matrix function and f is a regular distribution.

Corollary. *Let A and f be, respectively, a locally summable $l \times l$ matrix function* *and a locally summable l-vector function in* (α, β). *If* $\{\varphi_n\} \subset D^l_{+0}$, $\Delta\varphi_n = [a_n, b_n]$, $r \in R^l$ *are such that*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c \in (\alpha, \beta),$$

then

(i) *for n sufficiently large there is exactly one solution* x_n *of equation* (1) *satisfying the condition*

$$\langle x_n, \varphi_n\rangle = r,$$

(ii) *the sequence* $\{x_n\}$ *converges almost uniformly in* (α, β) *to a function x which is the solution of* (1) *satisfying the initial value problem* $x(c) = r$.

Proof. We can assume that $U(c) = I$. For n sufficiently large the assumptions on $\{a_n\}$ and $\{b_n\}$ imply (4) and this, together with Theorem 1, completes the proof of (i).

From (i) and Theorem 1, x_n are of the form

$$x_n(t) = U(t)C_n + U(t)g(t), \tag{11}$$

where

$$C_n = Q_n^{-1}(r - d_n) \quad (n = 1, 2, \ldots). \tag{12}$$

Just as in (7) and (9), Q_n and d_n are given by the formulae

$$d_n = \langle U(t)g(t), \varphi_n\rangle$$

and

$$Q_n \in H(\xi_1^n, \ldots, \xi_l^n) = \begin{bmatrix} u_{11}(\xi_1^n) & \ldots & u_{1l}(\xi_1^n) \\ \ldots & \ldots & \ldots \\ u_{l1}(\xi_l^n) & \ldots & u_{ll}(\xi_l^n) \end{bmatrix}, \tag{13}$$

$\xi_i^n \in \Delta\varphi_n$ $(i = 1, \ldots, l;\ n = 1, 2, \ldots)$.

In a similar way as (8) we can obtain

$$d_n = \int_\alpha^\beta \operatorname{diag}\varphi_n(t)\cdot U(t)g(t)\,dt = \begin{bmatrix} \sum_{j=1}^{l} u_{1j}(\eta_1^n)g_j(\eta_1^n) \\ \ldots \\ \sum_{j=1}^{l} u_{lj}(\eta_l^n)g_j(\eta_l^n) \end{bmatrix}, \tag{14}$$

where $g(t) = (g_1(t), \ldots, g_l(t))$, $\eta_i^n \in \Delta\varphi_n$ $(i = 1, \ldots, l;\ n = 1, 2, \ldots)$.

Since the matrix function $U(t)$ and the function $g(t)$ are continuous, conditions

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$$

and (13), (14) imply

$$\lim_{n\to\infty} d_n = g(c), \quad \lim_{n\to\infty} Q_n = Q = U(c) = I.$$

Hence, by (12), we have

$$\lim_{n\to\infty} C_n = C = r - g(c)$$

which, together with (11), completes the proof.

Theorem 2. *Suppose, as in Theorem* 1, *that* $A \in C_m^{l\times l}$, $f \in \tilde{D}_{m+1}^l$, $\varphi \in D_{+m}^l$, $r \in R^l$ *and*

$$\int_{\Delta\varphi} |A(t)|dt < \frac{\pi}{2}.$$

Let the sequences $\{A_n\} \subset C_m^{l\times l}$, $\{f_n\} \subset \tilde{D}_{m+1}^l$, $\{\varphi_n\} \subset D_{+m}^l$, $\{r_n\} \subset R^l$ *be convergent to* A, f, φ, r, *respectively. Then for* n *sufficiently large there is the unique solution* x_n *of the problem*

$$x' = A_n(t)x + f_n, \quad \langle x, \varphi_n\rangle = r_n, \tag{15}$$

and the sequence $\{x_n\} \subset \tilde{D}_m^l$ *converges to the unique solution* x *of problem* (1), (2).

Proof. The existence and uniqueness of solution of problem (1), (2) are the contents of Theorem 1. It is clear, by Remark 1, that for the existence and uniqueness of solution of problem (15) it is enough to show that for n sufficiently large the matrix

$$Q_n = \int_\alpha^\beta \operatorname{diag}\varphi_n(t)\cdot U_n(t)\,dt$$

is nonsingular ($U_n(t)$ being the fundamental matrix solution of system $x' = A_n(t)x$ and $U_n(c) = I$, where $c \in (\alpha, \beta)$, $n = 1, 2, \ldots$).

To this end we observe first that the assumption

$$\lim_{n\to\infty} A_n = A$$

implies that

$$\lim_{n\to\infty} U_n = U \quad \text{in} \quad C_{m+1}^{l\times l}\,(U(c) = I) \tag{16}$$

which, together with $\varphi_n \to \varphi$, gives

$$\lim Q_n = Q,$$

where

$$Q = \int_\alpha^\beta \operatorname{diag}\varphi(t)\cdot U(t)\,dt.$$

Hence and from the existence of Q^{-1} it follows that for n sufficiently large Q_n^{-1} also exists; moreover

$$\lim Q_n^{-1} = Q^{-1}.$$

In order to prove that $x_n \to x$ we note that

$$x_n = U_n(t)C_n + U_n(t)g_n, \tag{17}$$

where g_n in a primitive of the distribution $U_n^{-1}(t)f_n$ and

$$C_n = Q_n^{-1}(r_n - d_n).$$

The vector d_n has the form

(18) $$d_n = \langle U_n(t)g_n, \varphi_n\rangle .$$

Define the primitives g_n and g by

(19) $$\begin{aligned} \langle g_n, \psi\rangle &= -\langle U_n^{-1}(t)f_n, \psi^*\rangle, \\ \langle g, \psi\rangle &= -\langle U^{-1}(t)f, \psi^*\rangle, \end{aligned}$$

for every $\psi \in D_m$, where

$$\psi^*(t) = \int_a^t [\psi(s) - \varepsilon(s) \int_a^\beta \psi(\tau)d\tau]ds .$$

The function $\varepsilon \in D_{+m}$ is fixed.

By (16) it is easy to see that

$$\lim_{n\to\infty} U_n^{-1} = U^{-1} \text{ in } C_{n+1}^{l\times l} .$$

It is also clear that

$$\lim_{n\to\infty} U_n^{-1}(t)f_n = U^{-1}(t)f .$$

Hence, by (19), we have $g_n \to g$, and therefore

(20) $$\lim_{n\to\infty} U_n(t)g_n = U(t)g .$$

Conditions (18), (20) and the assumption $\varphi_n \to \varphi$ imply that $d_n \to d = \langle U(t)g, \varphi\rangle$. Now it is easily seen from the definition of C_n that

(21) $$\lim_{n\to\infty} C_n = C \quad \text{where} \quad C = Q^{-1}(r-d) .$$

Thus, by (17), conditions (16), (20), (21) yield immediately the assertion of Theorem 2.

REFERENCES

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