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Note on a Functional Equation

The object of the present note is the discussion of the functional equation

$$\Phi(z) = H\big(z, \Phi(f(z))\big), \tag{1}$$

where $\Phi(z)$ is the unknown function and $f(z)$ and $H(z, w)$ are complex-valued functions of complex variables.

We assume that f is analytic at $z = 0$ and

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad (|z| \leqslant r),$$

$$|a_1| < 1, \tag{2}$$

$H(z, w)$ is an analytic function of two complex variables (z, w) at the point $(0, 0)$ and $H(0, 0) = 0$.

Evidently, the necessary condition of the existence of an analytic solution of equation (1) is the existence of a formal solution.

In [1] the following theorem has been proved.

Every formal solution

$$\Phi(z) = \sum_{n=1}^{\infty} c_n z^n \tag{3}$$

of equation (1) *has a positive radius of convergence.*

In the present note we give a proof of this theorem, shorter than that in [1].

Suppose that (3) is a formal solution of equation (1). It follows from (2) that there exists a positive integer p such that

$$|H_w(0, 0)||f'(0)| < 1. \tag{4}$$

We may write

$$\Phi(z) = P(z) + z^p \varphi(z) \tag{5}$$

where

$$P(z) = \sum_{n=1}^{p-1} c_n z^n, \quad \varphi(z) = c_p + \sum_{n=p+1}^{\infty} c_n z^n.$$

We define the function

$$h(z, v) = \frac{H\big(z, P(f(z)) + (f(z))^p v\big) - P(z)}{z^p}. \tag{6}$$

We shall prove that $h(z, v)$ is analytic at $(0, \beta)$ for arbitrary β. Suppose that

$$H(z, w) = \sum_{n=0}^{\infty} a_n(z) w^n \quad (|z| \leqslant r,\ |w| \leqslant R).$$

Let us fix $R_0 > |\beta|$. Since $P(0) = 0$ and $f(0) = 0$, there exists $r_0 \leqslant r$ such that for $|z| \leqslant r_0$ and $|v| \leqslant R_0$ we have

$$|P(f(z)) + (f(z))^p v| \leqslant R.$$

Evidently, $f_1(z) = \dfrac{f(z)}{z}$ is analytic for $|z| \leqslant r_0$ and for $|z| \leqslant r_0$, $|v| \leqslant R_0$ we have

$$H\big(z, P(f(z)) + (f(z))^p v\big) - P(z) = \sum_{n=0}^{\infty} a_n(z)\big(P(f(z)) + z^p (f_1(z))^p v\big)^n - P(z) =$$

$$= \sum_{n=0}^{\infty} b_n(z) z^{np} v^n = b_0(z) + z^p \sum_{n=0}^{\infty} b_n(z) z^{p(n-1)} v^n,$$

where $b_n(z)$ are expressed by $a_n(z)$, $f(z)$, $f_1(z)$ and $P(z)$. Since the formal series $\varphi(z)$ formally satisfies the equation

$$z^p \varphi(z) = b_0(z) + z^p \sum_{n=1}^{\infty} b_n(z) z^{p(n-1)} \big(\varphi(f(z))\big)^n,$$

we infer that $b_0(z) = z^p b(z)$ where $b(z)$ is analytic for $|z| \leqslant r_0$. Thus $h(z, v)$ is analytic for $|z| \leqslant r_0$ and $|v| \leqslant R_0$.

By (6) we get

$$h'_v(z, v) = H'_w\big(z, P(f(z)) + (f(z))^p v\big) \left(\frac{f(z)}{z}\right)^p. \tag{7}$$

Moreover, $\varphi(z)$ formally satisfies the equation

$$\varphi(z) = h\big(z, \varphi(f(z))\big) \tag{8}$$

and consequently

$$c_p = h(0, c_p). \tag{9}$$

Equation (1) together with (5) is equivalent to (8). From (7) we obtain

$$h'_v(0, c_p) = H'_w(0, 0)(f'(0))^p.$$

Hence and from (4) there exist $\vartheta < 1$, $r_1 > 0$ and $R_1 > 0$ such that for $|z| \leqslant r_1$ and $|v_1 - c_p| \leqslant R_1$, $|v_2 - c_p| \leqslant R_1$ we have

$$|h(z, v_2) - h(z, v_2)| \leqslant \vartheta |v_2 - v_1|. \tag{10}$$

Let us fix $0 < K \leqslant R_1$. By the continuity of $h(z, v)$ there exists a number $r_2 > 0$ such that for $|z| \leqslant r_2$ we have

$$(11) \qquad |h(z, c_p) - h(0, c_p)| \leqslant (1-\vartheta)K.$$

Evidently, there exists an $r_3 > 0$ such that for $|z| \leqslant r_3$ we have

$$(12) \qquad |f(z)| \leqslant |z|.$$

Let $r = \min(r_1, r_2, r_3)$. We denote by A the set of all analytic functions $\varphi(z)$ which fulfil the condition

$$(13) \qquad |\varphi(z) - c_p| \leqslant K \quad (|z| \leqslant r, \ \varphi(0) = c_p).$$

This set with the metric $\varrho(\varphi_1, \varphi_2) = \sup\{|\varphi_2(z) - \varphi_1(z)| : |z| \leqslant r\}$ is a complete metric space.

Now we consider the transformation $\psi = T(\varphi)$ defined by the formula

$$(14) \qquad \psi(z) = h\big(z, \varphi(f(z))\big).$$

We shall prove that $T(A) \subset A$. If $\varphi \in A$, then from (14), (9), (12), (13), (10), and (11) we obtain for $|z| \leqslant r$

$$|\psi(z) - c_p| = |h\big(z, \varphi(f(z))\big) - h(0, c_p)| \leqslant |h\big(z, \varphi(f(z))\big) - h(z, c_p)| +$$
$$+ |h(z, c_p) - h(0, c_p)| \leqslant \vartheta|\varphi\big(f(z)\big) - c_p| + (1-\vartheta)K \leqslant \vartheta K + (1-\vartheta)K = K.$$

From (9) we get $\psi(0) = c_p$, so $\psi \in A$.
Let $\varphi_1, \varphi_2 \in A$, $\psi_1 = T(\varphi_1)$, $\psi_2 = T(\varphi_2)$. From (10) we get

$$\varrho(\psi_1, \psi_2) = \sup_{|z| \leqslant r} |h\big(z, \varphi_2(f(z))\big) - h\big(z, \varphi_1(f(z))\big)|$$
$$\leqslant \sup_{|z| \leqslant r} |\varphi_2\big(f(z)\big) - \varphi_1\big(f(z)\big)| \leqslant \vartheta \varrho(\varphi_1, \varphi_2).$$

Consequently T is a contraction. Hence, on account of the Banach contraction principle, it follows that there exists exactly one analytic solution $\varphi(z)$ of equation (8). Thus (5) is an analytic solution of equation (1).

REFERENCE

[1] W. Smajdor, *On the existence and uniqueness of analytic solutions of the functional equation* $\varphi(z) = h\big(z, \varphi(f(z))\big)$, Ann. Polon. Math. 19 (1967), 37—45.