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On Some Iterative-differential Equations III

The present paper is a continuation of the papers [3] and [4].

8. In the present section we give theorems concerning the global existence of solutions of the equation

$$(*) \qquad y(x) = c + \int_0^x f\big(t, y(t), y(y(t))\big)\,dt$$

and of a generalization of (*). We use the method given by T. Dłotko and M. Kuczma in [2] (for an equation of the type $y'(x) = f(x, y(x), y(h(x)))$, where h is a given function) with very little changes. The method of proving the compactness of a set in a functional space (the set V in the proof of Theorem 7) is given by A. Bielecki in [1] and used by the authors of [2].

Theorem 7. *Suppose that*

(a) f *is defined and continuous in*

$$D = \{(x, y_1, y_2) : 0 \leqslant x,\ 0 \leqslant y_1,\ y_2 < \infty\}$$

(b) *there exist positive constants* K, M, N, s *and a real-valued non-negative function* g *continuous in* $[0, s]$, *such that* $2N < M$ *and*

(i) $|f(x, y_1, y_2)| \leqslant K$ *for* $0 \leqslant y_j \leqslant \max(M, M-N+\max\{g(x) : 0 \leqslant x \leqslant s\})$

$$j = 1, 2$$

(ii) *for arbitrary real-valued functions* $y_j(x)$ $(j = 1, 2)$ *defined and continuous in* $[0, \infty)$ *fulfilling the inequalities*

$$0 \leqslant y_j(x) \leqslant h(x) \qquad (x \geqslant 0;\ j = 1, 2),$$

where $h(x) = g(x)+M-N$ *for* $x \in [0, s]$ *and* $h(x) = M$ *for* $x > s$ *we have*

$$-N \leqslant \int_0^x f\big(t, y_1(t), y_2(t)\big)\,dt \leqslant g(x) \quad \textit{for} \quad 0 \leqslant x \leqslant s,$$

$$\left|\int_0^x f\big(t, y_1(t), y_2(t)\big)\,dt\right| \leqslant N \textit{ for } x \geqslant s.$$

Under these assumptions, for every real number c such that $N \leqslant c \leqslant M-N$, there exists a solution of $(*)$ *of class C^1 on the whole half-axis $x \geqslant 0$.*

Proof. Let B be the linear space of real-valued functions u defined and continuous for $x \geqslant 0$, such that

$$\sup\{|u(x)|\exp(-x) : x \geqslant 0\} < \infty .$$

We put, for $u \in B$,

$$\|u\| = \sup\{|u(x)|\exp(-x) : x \geqslant 0\} . \tag{24}$$

The space B with the norm (24) is complete. We put

$$A = \{u \in B : 0 \leqslant u(x) \leqslant h(x) \quad \text{for} \quad x \geqslant 0\} . \tag{25}$$

It is easy to see that A is non-empty, convex, closed and bounded. Let $T: A \to B$ be defined as follows: for $u \in A$,

$$(Tu)(x) = c + \int_0^x f\big(t, u(t), u(u(t))\big)\,dt \quad \text{for} \quad x \geqslant 0 .$$

It is easy to show that Tu is continuous,

$$0 \leqslant (Tu)(x) \leqslant c+g(x) \quad \text{for} \quad 0 \leqslant x \leqslant s ,$$

$$0 \leqslant (Tu)(x) \leqslant c+N \quad \text{for} \quad x \geqslant s .$$

Hence, in virtue of the inequalities: $c+g(x) \leqslant M-N+g(x)$ and $c+N < M-N$ we have $T(A) \subset A$. We put $V = T(A)$. We shall prove that the transformation T is continuous with respect to the norm (24). Let $\{u_n\}$ be a sequence, such that $u_n \in A$, $u_n \to u$ as $n \to \infty$. We put $v_n = Tu_n$ and we shall prove that $v_n \to v = Tu$ as $n \to \infty$. We have

$$|v_n(x) - v(x)| \leqslant \int_0^x \Big|f\big(t, u_n(t), u_n(u_n(t))\big) - f\big(t, u(t), u(u(t))\big)\Big|\,dt \tag{26}$$

for $x \geqslant 0$, $n = 1, 2, \ldots$
Let λ be a positive number. In the interval $[0, \lambda]$, the sequence $\{u_n\}$ converges uniformly to u, so that for each $\eta > 0$ there exists a number $N(\eta, \lambda)$ such that $|u_n(x) - u(x)| < \eta$ for $n > N(\eta, \lambda)$ and for $x \in [0, \lambda]$. In particular, for each $\eta > 0$ there exists a number $N(\eta, \overline{M})$, such that $|u_n(t) - u(t)| < \eta$ for $n > N(\eta, \overline{M})$ and $t \in [0, \overline{M}]$, where $\overline{M} = \max(M, c + \max\{|g(x)| : 0 \leqslant x \leqslant s\})$. On the other hand, f being uniformly continuous in the set $[0, \lambda] \times [0, \overline{M}] \times [0, \overline{M}]$, for each $\varepsilon > 0$ there exists $\sigma > 0$, such that if $|y_j - \bar{y}_j| < \sigma$, $|y_j|, |\bar{y}_j| < \overline{M}$ $(j = 1, 2)$, then

$$|f(x, y_1, y_2) - f(x, \bar{y}_1, \bar{y}_2)| \leqslant \varepsilon \quad \text{for} \quad x \in [0, \lambda] .$$

Let $\varepsilon > 0$ be arbitrary. There exists $\sigma > 0$ such that if

$$|u_n(x) - u(x)| < \sigma , \tag{27}$$

$$(28) \qquad |u_n(u(x))-u(u(x))| < \sigma ,$$

$$(29) \qquad |u_n(u_n(x))-u_n(u(x))| < \sigma ,$$

then, for $x \in [0, \lambda]$,

$$(30) \qquad |f(x, u_n(x), u_n(u_n(x)))-f(x, u_n(x), u_n(u(x)))| \leqslant \frac{\varepsilon}{2} ,$$

$$(31) \qquad |f(x, u_n(x), u_n(u(x)))-f(x, u(x), u(u(x)))| \leqslant \frac{\varepsilon}{2} .$$

We can find $N_1 = N(\sigma, \lambda)$ such that (27) is fulfilled for $x \in [0, \lambda]$, $n > N_1$, and $N_2 = N(\sigma, \overline{M})$ such that (28) is fulfilled for $x \in [0, \lambda]$, $n > N_2$. Moreover, it is easy to see that

$$(32) \qquad |v_n(x)-v_n(t)| \leqslant K|x-t| \quad \text{for} \quad n = 1, 2, \ldots, \; x, t \geqslant 0$$

and that there exists $N_3 = N\left(\frac{\sigma}{K}, \lambda\right)$ such that (29) is fulfilled for $x \in [0, \lambda]$ and $n > N_3$. Thus for $x \in [0, \lambda]$ and $n > N_\varepsilon = \max(N_1, N_2, N_3)$ we have (30) and (31) and, in virtue of

$$(33) \quad |f(x, u_n(x), u_n(u_n(x)))-f(x, u(x), u(u(x)))| < \\ \leqslant |f(x, u_n(x), u_n(u_n(x)))-f(x, u_n(x), u_n(u(x)))|+ \\ +|f(x, u_n(x), u_n(u(x)))-f(x, u(x), u(u(x)))|$$

and (26),

$$(34) \qquad |v_n(x)-v(x)| \cdot \exp(-x) \leqslant \varepsilon\lambda \exp(-x) \quad \text{for} \quad x \in [0, \lambda], \; > n\, N_\varepsilon .$$

For $x \geqslant s$, we have $|v_n(x)-v(x)| \leqslant 2M$ and therefore, if $\lambda \geqslant s$,

$$(35) \qquad |v_n(x)-v(x)| \cdot \exp(-x) \leqslant 2M \exp(-x)$$

for $x \geqslant \lambda$ and each n.

Let μ be a positive number. There exists $\lambda \geqslant 0$ such that $2M\exp(-x) < \mu$ for $x \geqslant \lambda$. We can assume that $\lambda \geqslant s$. For this λ and for each $\varepsilon > 0$ we can find N (using the above procedure) such that, for $n > N$ and $x \in [0, \lambda]$,

$$(36) \qquad |v_n(x)-v(x)|\exp(-x) \leqslant \varepsilon\lambda\exp(-x) \leqslant \varepsilon\lambda .$$

Putting $\varepsilon = \mu/\lambda$ we obtain from (35) and (36), in virtue of $2M\exp(-x) < \mu$ for $x \geqslant \lambda$, the inequality

$$|v_n(x)-v(x)|\exp(-x) \leqslant \mu \quad \text{for} \quad n > N, \; x \geqslant 0 .$$

The proof of the continuity of the transformation T is finished.

Now, we will show, that V is compact. Let $\{v_n\}$ be a sequence of functions belonging to V. For each n, we have $|v_n(x)| \leqslant \overline{M}$ and (32). Hence, in virtue of the

Arzela theorem we can find a subsequence $\{v_n^1\}$ converging uniformly in $[0, 1]$ to a continuous function v^1. Similarly we can find a subsequence $\{v_n^2\}$ of $\{v_n^1\}$ which converges uniformly in $[0, 2]$ to a continuous function v^2. One can do this (by induction) for each m: we have a subsequence $\{v_n^m\}$ of the sequence $\{v_n^{m-1}\}$ which converges uniformly in $[0, m]$ to a continuous function v^m. Hence, the sequence $\{v_n^n\}$ converges uniformly in each closed subinterval of the interval $[0, \infty)$ to a function $v \in V$ such that $v = v^m$ in $[0, m]$. Let ε be an arbitrary positive number. We put $k = \ln 4\overline{M}\varepsilon^{-1}$. We have

$$\sup_{x\geqslant 0}|v(x)-v_n^n(x)|\exp(-x) \leqslant \sup_{0\leqslant x\leqslant k} |v(x)-v_n^n(x)|+2\overline{M}\exp(-k).$$

This means that if n is large enough then

$$||v-v_n^n|| \leqslant \varepsilon$$

and completes the proof of the compactness of V.

Now, using the fixed-point theorem of Schauder [5] and we obtain the existence of a solution of (*) as a fixed point of the transformation T. This solution is of course regular in $[0, \infty)$.

9. One can generalize Theorem 7 as follows

Theorem 8. *Suppose that*

(a) f *is a real-valued function continuous in*

$$G = \{(x, y_1, y_2): 0 \leqslant x, \ -\infty < y_1, y_2 < +\infty\}$$

(b) *there exist positive constants* K, M, N, s *and a real-valued function* g *continuous and non-negative in* $[0, s]$ *such that* $N < M$ *and*

(i) $|f(x, y_1, y_2)| \leqslant K$ *for* $|y_j| \leqslant \max\big(M, M-N+\max_{[0,s]}|g(x)|\big), \quad j = 1, 2,$ *and* $x \geqslant 0$,

(ii) *for arbitrary real-valued continuous functions* $y_j(x)$ $(j = 1, 2)$ *defined in* $[0, \infty)$ *fulfilling the inequalities*

$$|y_j(x)| \leqslant h(x) \quad (j = 1, 2), \ x \geqslant 0$$

where $h(x) = g(x)+M-N$ *for* $0 \leqslant x \leqslant s$ *and* $h(x) = M$ *for* $x > s$, *we have*

$$\Big|\int_0^x f\big(t, y_1(t), y_2(t)\big)dt\Big| \leqslant g(x) \quad \text{for} \quad x \in [0, s]$$

$$\Big|\int_0^x f\big(t, y_1(t), y_2(t)\big)dt\Big| \leqslant M \quad \text{for} \quad x \geqslant s.$$

Under these assumptions, for every real-valued function $c(x)$ *continuous in* $[-\overline{M}, 0]$, *where* $\overline{M} = \max\big(M, M-N+\max_{[0,s]}|g(x)|\big)$, *such that* $|c(0)| \leqslant M-N$ *there exists a solution* $y(x)$ *of the equation*

$$(**) \qquad y(x) = c(0)+\int_0^x f\Big(t, y(t), y\big(y(t)\big)\Big)dt,$$

continuous in $[-\overline{M}, \infty)$, *of class* C^1 *in* $[0, \infty)$, *fulfilling the condition*

(***) $$y(x) = c(x) \quad \textit{for} \quad x \in [-\overline{M}, 0].$$

The proof is quite similar to that of Theorem 7. Instead of A we consider

$$A' = \{u \in B : u(x)h(x) \text{ for } x \geqslant 0\},$$

where B is as in the proof of Theorem 7, and instead of the transformation T we consider T' given by the formula

$$(T'u)(x) = c(0) + \int_0^x f\big(t, \overline{u}(t), \overline{u}(\overline{u}(t))\big)\,dt \quad \text{for} \quad x \geqslant 0,$$

where $\overline{u}(x) = u(x)$ for $x \geqslant 0$ and $\overline{u}(x) = c(x)$ for $x \in [-\overline{M}, 0]$. It is easy to see that A' and T' have the same properties as A and T, and that one can use again the fixed-point theorem of Schauder.

10. Let $\hat{K}$, L, N be positive constants and let $\hat{a}$ be the solution of the equation $N\hat{a} = \exp(-L\hat{a})$. Let c be a number such that $|c| < \hat{a}$ and w a real-valued function continuous in $\langle 0, \infty)$ such that

(37) $$|w(x)| \leqslant |c| + x, \quad \text{for} \quad x \in \langle 0, \infty).$$

Theorem 9. *If* v *is a function continuous in* $[0, \infty)$ *such that*

(38) $$v(x) \leqslant L\int_0^x v(t)\,dt + N\int_0^x v\big(w(t)\big)\,dt \quad \textit{for} \quad x \geqslant 0,$$

then

(39) $$v(x) = 0 \quad \textit{for} \quad x \geqslant 0.$$

Proof. Let a be an arbitrary number from the open interval $(0, \hat{a})$. Hence, $Na < \exp(-La)$. Using the reasoning given in the proof of Theorem 1 in [3], for $K = 1$, we obtain $v(x) = 0$ for $x \in [0, a]$ and then, by the continuity of v,

(40) $$v(x) = 0 \quad \text{for} \quad x \in [0, \hat{a}].$$

Let $b_1 = \hat{a} - |c|$. Of course, $0 < b_1 \leqslant \hat{a}$ so that, by (37) and (40), we have $v(x) = v\big(w(x)\big) = 0$ for $x \in [0, b_1]$. Hence, for $x \in [b_1, \hat{a} + b_1]$,

$$v(x) \leqslant L\int_0^x v(t)\,dt + N\int_0^x v\big(w(t)\big)\,dt = L\int_{b_1}^x v(t)\,dt + N\int_{b_1}^x v\big(w(t)\big)\,dt$$

and, using the same reasoning as in the proof of the implication (38) $\Rightarrow$ (40), we obtain

$$v(x) = 0 \quad \text{for} \quad x \in [b_1, \hat{a} + b_1]$$

and then

$$v(x) = 0 \quad \text{for} \quad x \in [0, \hat{a} + b_1].$$

Putting $b_n = b_{n-1} - |c|$, we can show, by the same method, that $v(x) = 0$ for $x \in [0, b_n + a]$, and then $v(x) = 0$ for $x \geqslant 0$.

As a corollary of Theorem 9, we have the following

Theorem 10. *Suppose that M, N, c are constants such that $|c| < \hat{a}$, where $\hat{a}$ is the solution of the equation $N\hat{a} = \exp\big(-(M+N)\hat{a}\big)$. If $f = f(x, y, u)$ is defined in G (see Theorem 8) and*

$$|f(x, y, u) - f(x, z, t)| \leqslant M|y-z| + N|u-t|, \quad |f(x, y, u)| \leqslant 1$$

for every $(x, y, u), (x, z, t) \in G$, then there exists at most one function defined and continuous for $x \geqslant 0$ fulfilling $(*)$.

Proof. Let $y(x)$ and $z(x)$ be two continuous functions fulfilling $(*)$ for $x \geqslant 0$. It is easy to see that $|z(x)| \leqslant |c| + x$ for $x \geqslant 0$. We have

$$|y(x) - z(x)| \leqslant \int_0^x \Big(M|y(t) - z(t)| + N|y\big(y(t)\big) - z\big(z(t)\big)|\,dt\Big) <$$

$$\leqslant M\int_0^x |y(t)-z(t)|\,dt + N\int_0^x |y\big(y(t)\big) - y\big(z(t)\big)|\,dt + N\int_0^x |y\big(z(t)\big) - z\big(z(t)\big)|\,dt \leqslant$$

$$\leqslant (M+N)\int_0^x |y(t) - z(t)|\,dt + N\int_0^x |y\big(z(t)\big) - z\big(z(t)\big)|\,dt.$$

If we put $v(x) = |y(x) - z(x)|$ and $L = M+N$, $w(x) = z(x)$, then all assumptions of Theorem 9 will be satisfied and this will imply that $v(x) = 0$ for $x \geqslant 0$.

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