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Quasianalytic Functions of Several Complex Variables

INTRODUCTION

Let f be a real function continuous on a line segment $I=[a,b]$. Let $\mathcal{E}_\nu(f,I)$ denote the measure of the Tchebysheff best approximation to f on I by polynomials P_ν of degree $\leqslant \nu$,

$$\mathcal{E}_\nu(f,I)=\inf_{P\nu}\max_{x\in I}|f(x)-P_\nu(x)|.$$

Classification of continuous functions with the aid of the measure $\mathcal{E}_\nu(f,I)$ was started by Bernstein [1]. He found a connection between quickness of convergence to zero of the sequence $\{\mathcal{E}_\nu(f,I)\}$ for continuous functions belonging to a fixed class and their differentiability properties. Thus, for instance, the class $\mathcal{O}(I)$ of functions analytic on I can be characterized by the condition

$$\limsup_{\nu\to\infty}\sqrt[\nu]{\mathcal{E}_\nu(f,I)}<1.$$

It is well known that every function $f\in\mathcal{O}(I)$ vanishing on a subinterval $[\alpha,\beta]\subset I$ is necessarily equal to zero in I. Bernstein [1] has proved that this uniqueness principle holds true for the wider class of functions continuous on I and satisfying the inequality

$$\liminf_{\nu\to\infty}\sqrt[\nu]{\mathcal{E}_\nu(f,I)}<1$$

being called later *quasianalytic functions* in the Bernstein sense. Properties of these functions have been studied among others by Timan [12].

New classes of quasianalytic functions were introduced by means of the measure $\mathcal{E}_\nu(f,I)$ by Mergelyan [5].

Recently Mamedov [3] has defined *strong quasianalytic functions* in the class $\mathcal{L}_p(D,\mu)$ of all functions of n real variables defined in the cube $D=\{(x_1,\ldots,x_n)\in R^n: |x_k|\leqslant 1, k=1,\ldots,n\}$, with the p-th powers of their absolute values Lebesgue integrable with weight $\mu>0$.

In this paper we define quasianalytic functions in the Bernstein sense in the space $\mathcal{C}(I)$ of functions of n complex variables continuous in a compact set $I \subset C^n$ (section 2) and we prove some properties of them analogous to well known properties of quasianalytic functions introduced by Bernstein.

For instance, we give sufficient conditions for compact sets I and E, $E \subset I$, in the space C^n that every function quasianalytic on I vanishing in E be equal to zero in I (theorems 3 and 5). In the case $n = 1$, if I is a line segment, those theorems generalize the already quoted result of Bernstein and a certain result of Szmuszkowiczówna [11].

In the last section we give examples of quasianalytic classes $Q(I)$ of functions defined in $I \subset C^n$, quasianalytic in the sense that every two functions $f, g \in Q(I)$ equal in a subset E of I are identical on I.

Methods of proving used in this paper are based on properties of some extremal function $\Phi(z, I)$ of a compact set $I \subset C^n$ introduced by Siciak [7] and on his results establishing connections of $\Phi(z, I)$ with the approximation theory in C^n (see section 1).

1. THE EXTREMAL FUNCTION $\Phi(z, E)$

Let E be a compact set in the space C^n of n complex variables $z = (z_1, \ldots, z_n)$. Let $\mathcal{C}(E)$ denote the Banach space of complex functions continuous on E with the norm

$$||f||_E = \max_{z \in E} |f(z)|.$$

Let $\mathcal{M}_\nu$ denote the family of all polynomials $P(z)$ of n complex variables $z = (z_1, \ldots, z_n)$,

$$P(z) = \sum_{k_1 + \ldots + k_n \leqslant \nu} a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n},$$

and let $\mathcal{M}$ denote the sum $\bigcup_{\nu=0}^{\infty} \mathcal{M}_\nu$. The function

$$\Phi(z, E) = \sup_{\nu \geqslant 1} [\sup\{|P(z)|^{1/\nu} : P \in \mathcal{M}_\nu,\ ||P||_E \leqslant 1\}] \tag{1.1}$$

is the extremal function of E introduced by Siciak ([7], theorem 2, p. 335; see also [8]). In the case $n = 1$, function (1.1) is equal to the Leja's extremal function (see [2], p. 265) introduced in another way.

We shall often use the following properties of $\Phi(z, E)$ ([7])

$$\Phi(z, E) = 1, \quad z \in E; \tag{1.2}$$

$$\Phi(z, E) \geqslant 1, \quad z \in C^n; \tag{1.3}$$

$$\Phi(z, E) \leqslant \Phi(z, F), \quad z \in C^n, \tag{1.4}$$

if F is a closed subset of E;

$$|P(z)| \leqslant ||P||_E \Phi^\nu(z, E), \quad z \in C^n, \tag{1.5}$$

for any polynomial $P \in \mathcal{M}_\nu$, $\nu = 0, 1, \ldots$

Lemma 1 ([7]). *If $E = E_1 \times \ldots \times E_n$, where E_i denotes a compact set of positive transfinite diameter $d(E_i)$ in the complex z_i-plane, then*

$$\Phi(z, E) = \max_{1 \leqslant i \leqslant n} \{\Phi(z_i, E_i)\}, \quad z \in C^n.$$

In the sequel we shall always assume that *continuum* is a compact connected set not reduced to a point. It follows from lemma 1, that if E_i, $i = 1, \ldots, n$, is a continuum then the function $\Phi(z, E_1 \times \ldots \times E_n)$ is continuous in C^n.

Let $\mathcal{E}_\nu(f, E)$ denote the measure of the Tchebysheff best approximation to $f \in \mathcal{C}(E)$ on E by polynomials $P \in \mathcal{M}_\nu$,

$$\mathcal{E}_\nu(f, E) = \inf_{P \in \mathcal{M}_\nu} ||f - P||_E.$$

Lemma 2. *Let $E \subset C^n$ be a compact set such that $\Phi(z, E)$ is continuous in E. Let f be a bounded function defined on E. Then:*

1° *If*

$$\limsup_{\nu \to \infty} \sqrt[\nu]{\mathcal{E}_\nu(f, E)} < 1, \tag{1.6}$$

then there exists a function $\tilde{f}$ holomorphic in an open neighbourhood G of E such that $\tilde{f} = f$ in E ([7]).

2° *If E is polynomially convex then* (1.6) *is a necessary and sufficient condition that the function f have a holomorphic extension $\tilde{f}$ to an open neighbourhood of E* ([10]).

2. QUASIANALYTIC FUNCTIONS IN THE BERNSTEIN SENSE

Let E be a compact set in the space C^n.

Definition 1. We shall say that a function $f \in \mathcal{C}(E)$ is *quasianalytic* on E *in the Bernstein sense* if

$$\liminf_{\nu \to \infty} \sqrt[\nu]{\mathcal{E}_\nu(f, E)} < 1. \tag{2.1}$$

The class of all functions $f \in \mathcal{C}(E)$ satisfying (2.1) will be denoted by $\mathcal{B}(E)$.

Given any $\varrho \in (0, 1]$ we define a subclass $\mathcal{B}_\varrho(E)$ of $\mathcal{B}(E)$ by the inequality

$$\liminf_{\nu \to \infty} \sqrt[\nu]{\mathcal{E}_\nu(f, E)} < \varrho. \tag{2.2}$$

Let $\mathcal{P}(E)$ denote the Banach space of functions uniformly approximated on E by polynomials $P \in \mathcal{M}$.

Theorem 1. *For every function* $f \in \mathfrak{T}(E)$ *and for every* $\varrho \in (0, 1]$ *there exist functions* $f_1, f_2 \in \mathfrak{B}_\varrho(E)$ *such that* $f = f_1 + f_2$ *on* E.

Proof. Given any function $f \in \mathfrak{T}(E)$ and a number $\varrho \in (0, 1]$, we define a sequence $\{\nu_k\}$ of positive integers and a sequence of polynomials $\{P_{\nu_k}\} \subset \mathcal{M}$ as follows

$$P_{\nu_0}(z) \equiv 0,$$

$$\nu_{k+1} > \nu_k, \quad k = 0, 1, \ldots$$

$$\mathcal{E}_{\nu_{k+1}}(f, E) = \|f - P_{\nu_{k+1}}\|_E \leqslant \varrho_1^{\nu_k}/2,$$

where $\varrho_1 \in (0, \varrho)$. Put

$$f_1(z) = \sum_{k=1}^{\infty} [P_{\nu_{2k}}(z) - P_{\nu_{2k-1}}(z)],$$

and

$$f_2(z) = \sum_{k=1}^{\infty} [P_{\nu_{2k-1}}(z) - P_{\nu_{2k-2}}(z)].$$

It is clear that $f_1(z) + f_2(z) = f(z)$, $z \in E$. We have also

$$\mathcal{E}_{\nu_{2l}}(f_1, E) \leqslant \Big\| \sum_{k=l+1}^{\infty} (P_{\nu_{2k}} - P_{\nu_{2k-1}}) \Big\|_E \leqslant \sum_{k=l+1}^{\infty} 2\mathcal{E}_{\nu_{2k-1}}(f, E) \leqslant \sum_{k=l+1}^{\infty} \varrho_1^{\nu_{2k-2}} \leqslant M\varrho_1^{\nu_{2l}},$$

and

$$\mathcal{E}_{\nu_{2l-1}}(f_2, E) \leqslant \Big\| \sum_{k=l+1}^{\infty} (P_{\nu_{2k-1}} - P_{\nu_{2k-2}}) \Big\|_E \leqslant \sum_{k=l+1}^{\infty} 2E_{\nu_{2k-2}}(f, E) \leqslant \sum_{k=l+1}^{\infty} \varrho_1^{\nu_{2k-3}} \leqslant M\varrho_1^{\nu_{2l-1}},$$

M being a constant. Hence, according to (2.2), $f_1, f_2 \in \mathfrak{B}_\varrho(E)$.

Let E be a compact set in R^n. It follows from the well known theorem of Weierstrass that $\mathfrak{T}(E) = \mathfrak{C}(E)$ in this case. Thus, we have the following

Corollary. *If* E *is a compact set in* R^n *then theorem* 1 *holds for every function* $f \in \mathfrak{C}(E)$.

In the case $E = [a, b] \subset R$, theorem 1 is the well known result of Markushevich [4] (see also [12], p. 388). The idea of the proof given above is due to Mergelyan [6]. Mamedov [3] has recently announced that analogous theorem holds for functions for the class $\mathfrak{L}_p(D, \mu)$.

3. UNIQUENESS THEOREMS FOR QUASIANALYTIC FUNCTIONS

Let $\mathcal{F}(X)$ denote any family of functions defined in a space X and let Z be a subset of X. We shall say that Z is a *nullifying subset* of X for the family $\mathcal{F}(X)$ (we shall write $Z \in N_{\mathcal{F}(X)}$) if

$$f(x) = 0, \ x \in Z \Rightarrow f(x) = 0, \ x \in X.$$

Further on, given a condition (W), for a set E satisfying (W) we shall write $E \in (W)$.

Let now I be a compact set in C^n. If E is a compact subset of I we shall use the following

Condition (W_1). $E \in (W_1)$, if the function $\Phi(z, E)$ is bounded on I.

It is known ([2], p. 265), in the case $n = 1$, that if $d(E) > 0$ then the function $\Phi(z, E)$ is bounded on every compact set in the complex plane C. Hence, by (1.4) and lemma 1, if the set E, $E \subset I$, contains a compact subset $F = F_1 \times \ldots \times F_n$, where $d(F_i) > 0$, then $E \in (W_1)$.

Theorem 2. *If* $E \in (W_1)$ *then* $E \in N_{\mathfrak{B}_t(I)}$, *where* $t = 1/\sup\limits_{z \in I} \Phi(z, E)$.

Proof. Let $f \in \mathfrak{B}_t(I)$ and $f(z) = 0$ for $z \in E$. Let P_ν denote a polynomial of the Tchebysheff best approximation to f on I of order ν, $\nu = 0, 1, \ldots$ It follows from (2.2) that there exist two positive constants $M = M(f)$ and $\varrho = \varrho(f)$, $\varrho \in (0, t)$ such that for an increasing sequence $\{\nu_k\}$ of positive integers we have

$$||P_{\nu_k}||_E = ||f - P_{\nu_k}||_E \leqslant ||f - P_{\nu_k}||_I \leqslant M\varrho^{\nu_k}.$$

Therefore, by (1.5),

$$|P_{\nu_k}(z)| \leqslant M\varrho^{\nu_k}\Phi^{\nu_k}(z, E) \leqslant M(\varrho/t)^{\nu_k}, \quad z \in I,$$

whence

$$\liminf_{\nu \to \infty} P_\nu(z) = 0, \quad z \in I.$$

Thus, by (2.2), $f(z) = 0$, $z \in I$.

Now consider the following problem: what conditions should be fulfiled by the sets I and E, $E \subset I$, in order that every function $f \in \mathfrak{B}(I)$ vanishing on E be equal to zero in I. A partial answer to this question is given by the following

Theorem 3. *If* $I = I_1 \times \ldots \times I_n$ *and* $I \supset E = E_1 \times \ldots \times E_n$, *where* I_i, E_i *are continua in the complex* z_i*-plane, then* $E \in N_{\mathfrak{B}(I)}$.

Proof. Let $f \in \mathfrak{B}(I)$ and $f(z) = 0$ for $z \in E$. According to (2.1), there exist two positive constants $M = M(f)$ and $\varrho = \varrho(f)$, $\varrho \in (0, 1)$, such that

$$\mathcal{E}_{\nu_k}(f, I) \leqslant M\varrho^{\nu_k}, \tag{3.1}$$

$\{\nu_k\}$ being an increasing sequence of positive integers. If P_{ν_k}, $k = 1, 2, \ldots$, denote polynomials such that $\mathcal{E}_{\nu_k}(f, I) = ||f - P_{\nu_k}||_I$, we have by assumption

$$||P_{\nu_k}||_E = ||f - P_{\nu_k}||_E \leqslant ||f - P_{\nu_k}||_I \leqslant M\varrho^{\nu_k}.$$

Hence, in virtue of (1.5),

$$|P_{\nu_k}(z)| \leqslant M[\varrho\Phi(z, E)]^{\nu_k}.$$

Therefore, by lemma 1, we have

$$|P_{\nu_k}(z)| \leqslant M[\varrho \max_i \{\Phi(z_i, E_i)\}]^{\nu_k}, \tag{3.2}$$

where the functions $\Phi(z_i, E_i)$, $i = 1, \ldots, n$, are continuous in the complex z_i-plane, respectively. Hence, for any $\eta \in (\varrho, 1)$, by (1.2) and (1.3), there exist open sets $U_i(\eta)$ such that $E_i \subset U_i(\eta)$ and

$$\varrho\Phi(z_i, E_i) < \eta, \quad z_i \in U_i(\eta), \quad i = 1, \ldots, n.$$

Without loss of generality we may consider only the case $i = 1$. Now, choosing a continuum $F_1 \subset I_1$, $F_1 \neq E_1$, such that $E_1 \subset F_1 \subset U_1(\eta)$, by (3.2) we have

$$|P(_{\nu_k}z)| \leqslant M\eta^{\nu_k}, \quad z \in F_1 \times E_2 \times \ldots \times E_n,$$

whence, by (3.1),

$$f(z) = 0, \quad z \in F_1 \times E_1 \times \ldots \times E_n.$$

Let $\{F_1^t\}_{t\in T}$ be a family of all the continua containing the set E_1 and such that

$$f(z) = 0 \quad \text{for} \quad z \in F_1^t \times E_2 \times \ldots \times E_n, \quad t \in T.$$

Denote by S_1 the sum $\bigcup_{t\in T} F_1^t$. By the continuity of f, we have

$$f(z) = 0, \quad z \in \overline{S}_1 \times E_2 \times \ldots \times E_n.$$

It is obvious that $\overline{S}_1 \in \{F_1^t\}_{t\in T}$. Suppose $\overline{S}_1 \neq I_1$. Then, by an analogous reasoning as given above, we can find a continuum H_1, $H_1 \neq \overline{S}_1$, such that $\overline{S}_1 \subset H_1 \subset I_1$ and

$$f(z) = 0, \quad z \in H_1 \times E_2 \times \ldots \times E_n,$$

what is in contradiction to the construction of S_1. Thus $\overline{S}_1 = I_1$ and the proof is completed.

In the case $n = 1$, if sets I and E reduce to line segments, theorem 3 is the well known result of Bernstein [1] (see also [12], p. 386). If $I = \{(x_1, \ldots, x_n) \in R^n : |x_i| \leqslant 1, i = 1, \ldots, n\}$ and $E = \{(x_1, \ldots, x_n) \in I : |x_i| \leqslant 1-\varepsilon, i = 1, \ldots, n, \varepsilon > 0\}$, theorem 3 is equivalent to a result of Mamedov [3] for strong quasianalytic functions in the class $\mathfrak{L}_p(I, \mu)$, $p = \infty$.

Now, we shall generalize theorem 3. For this purpose we shall formulate some conditions.

Condition (W_2). Let E be a compact set in C^n. $E \in (W_2)$ if there exists a point $a \in E$ such that the function $\Phi(z, E)$ is continuous at a.

Polynomial condition (L) (see [9]). $E \in (L)$ at a point $a \in C^n$ if for every family $\mathcal{A}(E) \subset \mathcal{M}$ of polynomials uniformly bounded on E and for every $\varepsilon > 0$ there exist two positive numbers $M = M(a, \varepsilon)$ and $\delta = \delta(a, \varepsilon)$ such that

$$|P(z)| \leqslant Me^{\varepsilon\nu}, \quad \operatorname{dist}(z, a) < \delta, \quad P \in \mathcal{A}(E),$$

where ν is the degree of P.

By the definition of the function $\Phi(z, E)$, if $E \in (L)$ at a point $a \in E$ then $E \in (W_2)$.

Theorem 4. *If $E = E_1 \times \ldots \times E_n$ and $d(E_i) > 0$, $i = 1, \ldots, n$, then $E \in (W_2)$.*

Proof. By our assumption and by lemma 1, we have

$$\Phi(z, E) = \max_{1\leqslant i\leqslant n} \{\Phi(z_i, E_i)\}.$$

It is known ([7], [8]) that the function $G(z_i) = \log\Phi(z_i, E_i)$ is a generalized Green's function for the unbounded component $D_\infty(E_i)$ of $C - E_i$ with pole at ∞. Therefore

$$(3.3) \qquad \Phi(z, E) = \max_{1 \leqslant i \leqslant n} \{\exp G(z_i)\}, \quad z \in D_\infty(E_1) \times \ldots \times D_\infty(E_n).$$

Since $d(E_i) > 0$, by the well known lemma of Kellog there exists a point $\mathring{z}_i \in E_i$ regular with respect to the function $G(z_i)$. Hence and by (3.3), the function $\Phi(z, E)$ is continuous at $\mathring{z} = (\mathring{z}_1, \ldots, \mathring{z}_n) \in E$.

Let $\mathfrak{R}$ denote a family of bounded sets in C^n such that every point z of a set $I \in \mathfrak{R}$ belongs to I with a Cartesian product $C_1 \times \ldots \times C_n$ of continua C_i in the complex z_i-plane, respectively. For every set $I \in \mathfrak{R}$ we can define the family $\mathcal{F}(I) = \{F_t\}_{t \in T}$ of all the subsets of I such that

$$F_t = F_1^t \times \ldots \times F_n^t, \quad t \in T,$$

where F_i^t, $i = 1, \ldots, n$, are continua in the complex z_i-plane. The family $\mathcal{F}(I)$ has obviously the following properties

1° for every point $z \in I$ there exists an index $t = t(z) \in T$ such that $z \in \mathcal{F}_t \in \mathcal{F}(I)$,

2° $I = \bigcup\limits_{t \in T} F_t$.

For sets $I \in \mathfrak{R}$ we formulate the following

Condition (W_3). $I \in (W_3)$ if the family $\mathcal{F}(I)$ has the following property: for every subset T_1 of the index set T, $T_1 \neq \emptyset$, $T_1 \neq T$, there exists an index $t_0 \in T - T_1$ such that

$$F_{t_0} \cap \bigcup_{t \in T_1} F_t \neq \emptyset.$$

It is clear that every set satisfying (W_3) is connected.

We shall now prove generalization of theorem 3.

Theorem 5. *If $I \in (W_3)$ and $E \in (W_2)$, $E \subset \bar{I}$, then $E \in N_{\mathfrak{B}(\bar{I})}$.*

Proof. Let $f \in \mathfrak{B}(\bar{I})$ and $f(z) = 0$, $z \in E$. Let P_ν denote a polynomial such that $\mathcal{E}_\nu(f, I) = \|f - P_\nu\|_{\bar{I}}$, $\nu = 0, 1, \ldots$ According to definition 1 and in view of (1.5), for an increasing sequence $\{\nu_k\}$ of positive integers we have

$$(3.4) \qquad |P_{\nu_k}(z)| \leqslant M[\varrho\Phi(z, E)]^{\nu_k},$$

M and ϱ, $\varrho \in (0, 1)$, being positive constants depending on f. Since $E \in (W_2)$ for a point $a = (a_1, \ldots, a_n) \in E$ and for any $\eta \in (\varrho, 1)$ there exists a polydisc $P(a, r) = K(a_1, r_1) \times \ldots \times K(a_n, r_n)$, where $K(a_i, r_i) = \{z_i \in C : |z_i - a_i| \leqslant r_i\}$, $i = 1, 2, \ldots, n$, such that

$$\varrho\Phi(z, E) < \eta, \quad z \in P(a, r).$$

Hence, by (3.4),

$$|P_{\nu_k}(z)| \leqslant M\eta^{\nu_k}, \quad z \in P(a, r).$$

Therefore, by (2.1), we have

$$(3.5) \qquad f(z) = 0, \quad z \in P(a, r) \cap \bar{I}.$$